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SHELLS OF POSITIVE GAUSSIAN CURVATURE UNDER POSTCRITICAL DEFORMATIONS

II. Loss of Stability

TRANSLATED FROM RUSSIAN

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(Strogo vypuklye obolochki pri zakriticheskikh deformatsiyakh)

II LOSS OF STABILITY

(II. Poterya ustoichivosti obolochek)

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In this book a new method of investigating the problem of shells of positive Gaussian curvature is presented. The critical loads are determined for various modes of loading.

The book is of interest to designers, scientific workers, students, and Ph. D. candidates.

INTRODUCTION

This book investigates the stability of shells of positive Gaussian curvature when subjected to various modes of loading. The method used is based on the following two considerations:

1. The load supported by the shell at the moment of loss of stability is stationary and, therefore, changes but little under a substantial bulging of the shell.
2. The deformation of the shell under substantial bulging beyond the neighborhood of the boundary of the bulging zone can be considered as a geometric bending.

It follows from condition 2 that the energy of deformation of the shell at the primary stage of postcritical deformation is concentrated, in the main, along the boundary of the zone (zones) of bulging. In section 1 the following formula for the energy of deformation, per unit length of boundary, is derived

$$\bar{U} = \frac{2E\delta^2\alpha^2 h}{\sqrt{12}(1-\nu^2)\rho}.$$

Here, α is the angle between the plane of contact to the boundary of the bulging zone, γ, and the planes tangent to the surface; ρ is the radius of curvature of γ; h, the change in shell deflection when passing across the boundary into the zone of bulging; δ, the thickness of the shell; E, the modulus of elasticity; and ν, Poisson's ratio. The establishment of this formula is the basic result of this section, and is widely used in all further deliberations.

In the same section a study is made of the problem of loss of stability of a shallow shell rigidly fixed at the edge and subjected to uniform external pressure. A formula for the value of the critical pressure at which loss of stability occurs,

$$p = \frac{2E\delta^2}{\sqrt{3(1-\nu^2)}\, R_1 R_2},$$

is derived. Here, R_1 and R_2 denote the principal radii of curvature at the center of bulging.

At the end of the section the problem of loss of stability of a shell subjected to the pressure of a tightly drawn string is considered. The following formula for the critical pull of the string, Q, is derived

$$Q = 3\pi c E\delta^{5/2}\left(\frac{R_1+R_2}{R_1R_2}\right)\sqrt{R_n},$$

where R_n is the radius of normal curvature of the shell surface in the direction of the string, and c is a constant $\simeq 0.2$. The question of loss of stability of a shell supported by an elastic foundation is studied and a formula for the critical stress is obtained.

In section 2 a study is made of the geometry of the primary stage of the postcritical deformation of the shell. Starting with the assumption that change of shape of the shell beyond the neighborhood of the bulging zone boundary is small, we pass from finite bending to the infinitely small and derive an explicit expression for the bending field. The study is confined to the case when the zone of bulging is small, has an elliptic shape, and is freely orientated with respect to the principal directions of the shell surface. Results arrived at in this section find substantial application later on.

The third section begins again with a study of the problem of critical external pressure for shallow shells of positive Gaussian curvature and a rigidly fixed edge. This study is different from that of section 1 in that the deformations considered are general. Such an approach frees us of a number of limitations which narrowed down the application range of the results of section 1. The formula for the value of the critical pressure, arrived at in this section, is identical with the previous result, namely

$$p = \frac{2E\delta^2}{\sqrt{3}\,(1-\nu^2)\,R_1R_2}.$$

Section 3 also includes a discussion of the question of internal and external critical pressure for a shell of rotation of positive Gaussian curvature. It is found that internal critical pressure for such a shell is given by the formula

$$p = \frac{2E\delta^2}{\sqrt{3}\,(1-\nu^2)\,R_1R_2} \cdot \frac{1}{\frac{\rho^2}{2R_1R_2} - 1},$$

where ρ is the radius of the parallel along which the regions of bulging are situated; all other quantities retain their previous meanings and are related to the centers of bulging. It is shown that the regions of bulging are strongly elongated along the meridians of the surface.

The value of the external critical pressure for a convex shell of rotation is given by the following formula:

$$p = \frac{2E\delta^2}{\sqrt{3}(1-\nu^2)\,R_1R_2} \cdot \frac{1}{\frac{\rho^2}{2R_1R_2} + 1}.$$

When applied to the case of a closed spherical shell, this formula gives the following value for the critical pressure:

$$p = \frac{2E\delta^2}{\sqrt{3}\,(1-\nu^2)\,R^2}\,\frac{2}{3},$$

which equals $\frac{2}{3}$ of the value obtained for a shallow shell. Loss of stability is accompanied by the formation of flattened out dents along the equator. Application of the general result to the case of shallow shells does not contradict the corresponding formulas for shallow shells because of smallness of the ratio $\rho^2/2R_1R_2$. In addition, it agrees with known experimental facts according to which, in a number of cases, loss of stability of a convex shell under external pressure is accompanied by the formation of a system of bulging regions along the edge of the shell.

Section 3 also includes a study of loss of stability of shells of rotation subjected to twist. For the value of the critical moment causing loss of stability, the following formula is obtained:

$$M=\frac{2\pi\rho^2E\delta^2}{\sqrt{3(1-\nu^2)}\sqrt{R_1R_2}},$$

where ρ is the radius of the parallel along which bulging of the shell takes place.

The fourth section studies postcritical deformations proper of a shell of positive Gaussian curvature subjected to two modes of loading: a concentrated load, and uniform external pressure. This problem was discussed previously in /2/. The innovation here consists in the derivation of a more exact expression for the shell deformation energy. Results arrived at are used to study the influence of initial bending of the shell on the critical load. It is shown that the value of the critical load for a shallow shell with edge rigidly fixed and subjected to external pressure is decreased and is equal to

$$p=\frac{3}{2}c\frac{1}{\sqrt{R_1R_2}}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\delta^2\sqrt{\frac{\delta}{2h}},$$

where $2h$ is the initial deflection. This formula was derived under the assumption of substantial initial deflection. In any case, it is necessary to assume that $2h/\delta>1$.

A method of approach to the study of the stability problem of a three-layer shell is presented in the supplement to this work. As an example, the value of the critical pressure for a shallow shell of positive Gaussian curvature is obtained:

$$p=\frac{2E\delta^2}{\sqrt{3(1-\nu^2)}R_1R_2}2+\frac{Gt(R_1+R_2)}{R_1R_2},$$

where δ is the thickness of the outer layers; t, the thickness of the inner layer; E, the modulus of elasticity of the outer layers; and G, the shear modulus of the filler.

§1. LOSS OF STABILITY OF A SHELL OF POSITIVE GAUSSIAN CURVATURE WHEN SUBJECTED TO EXTERNAL PRESSURE

We showed in /1/ that our method for studying postcritical deformations can be used to investigate stability problems. We also derived a formula for the value of the upper critical load for a shallow spherical segment subjected to uniform external pressure. The same considerations were taken into account for this derivation as were used in the study of post-critical deformations. In this section this method will be applied to a general study of shallow shells of positive Gaussian curvature. At first, we shall confine ourselves to simple problems in order to apply this new method to simple examples with well-known results.

1. Energy of elastic deformation of a shell at the initial stage of bulging

We shall consider a shallow shell of positive Gaussian curvature, fixed at its edges and subjected to external pressure. For a certain value of such a pressure the shell will lose its stability. It was shown in /1/, a study of postcritical deformation of a spherical shell after loss of stability, that in the case of a uniform load such a deformation must start by bulging occurring throughout a certain region. The case when bulging begins to spread out from some central point is excluded. All the above, taken in conjunction with the fact that in the final phase the deformation of the shell under consideration must approximate to the state of mirror image bulging, enables us to regard the shape of the shell at the starting phase of post-critical deformation as being close enough to the state of double mirror image bulging (Figure 1). It is assumed, of course, that the degree of fixity of the shell edge at the support is rigid and that the bulging region comprises a sizeable portion of the shell, thus enabling the rigidity of the fixed edge to predetermine the above-mentioned shell deformation at the final phase.

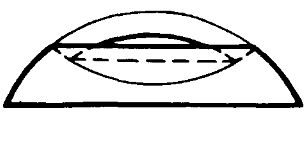

FIGURE 1.

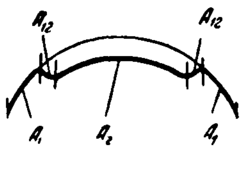

FIGURE 2.

In § 3 we shall examine a more general case of postcritical deformation at the initial phase, relaxing the requirement regarding mode of edge support and size of bulging region. However, as we shall see presently, this will not alter the final value of the magnitude of critical pressure obtained in the present problem. We shall therefore limit ourselves in the present section to the above-mentioned simple deformation and will concentrate our attention on other questions related to the applicability of our method, e.g., the evaluation of shell deformation energy, which we will deal with presently.

Starting with our initial assumption regarding the character of postcritical deformation of the shell surface we consider that the deformed shape of the shell approximates well to its original shape within zone A_1 (Figure 2) and to the double mirror image bulging within zone A_2. The transitive zone A_{12} is considered narrow enough. Let γ denote the assumed curve separating zones A_1 and A_2, ρ, its radius of curvature, and α, the angle between the plane of γ and the planes tangent to the surface.

Using the procedure of Appendix II of /1/, where the value of the deformation energy of a spherical shell was derived, we obtain for the deformation energy U in the zone of substantial local bending of the shell (A_{12}), and for unit length of γ, the expression

$$\overline{U} = \frac{\delta E}{2(1-\mu^2)} \int\limits_{-\varepsilon^*}^{\varepsilon^*} \left(\frac{\delta^2 v''^2}{12} + \frac{u^2}{\rho^2} \right) ds.$$

In the above expression v denotes the normal displacement of points of the shell middle surface under deformation, and u, the displacement in the tangent plane in a direction perpendicular to γ. Integration between the limits $-\varepsilon^*$, ε^* is carried out along the assumed breadth of zone A_{12}. Ignoring deformation of the middle surface in a direction perpendicular to γ, we arrive at the well-known expression relating the displacements u and v,

$$u' + \alpha v' + \frac{1}{2} v'^2 = 0.$$

Remarks. In the present discussion, as well as previously, whenever we speak of the energy of deformation in the zone of strong local bending we consider, in addition to shell bending, the accompanying extension (contraction) of its middle surface. Of the two terms in the expression under the integral sign, the first accounts for the energy of bending and the second for the energy of extension (contraction) of the middle surface.

As in our previous studies, we replace the variables u, v, s by variables $\bar{u}$, $\bar{v}$, $\bar{s}$, where

$$\bar{u} = \frac{u}{\varepsilon \rho \alpha^2}, \quad \bar{v} = \frac{v'}{\alpha}, \quad \bar{s} = \frac{s}{\rho \varepsilon},$$
$$\varepsilon^4 = \frac{\delta^2}{12 \rho^2 \alpha^2}.$$

Omitting, for simplicity, the bars in these new variables, the expression for $\overline{U}$ will take the form:

$$\overline{U} = \frac{E \delta^{5/2} \alpha^{5/2} \rho^{-1/2}}{2 \cdot 12^{3/4} (1-\mu^2)} \int\limits_{-\bar{\varepsilon}^*}^{\bar{\varepsilon}^*} (v'^2 + u^2)\, ds.$$

The limits of integration $\bar{\varepsilon}^*$ and $-\bar{\varepsilon}^*$ increase indefinitely in their absolute value as $\rho\alpha/\delta$ increases. Therefore, upon limiting ourselves to the case of such shells and deformations for which $\delta/\rho\alpha$ is small, the limits of integration can be changed to $\pm\infty$. Then

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{2\cdot 12^{3/4}(1-\mu^2)}\int_{-\infty}^{\infty}(v'^2+u^2)\,ds.$$

We shall always take the function $(v)s$ to be symmetric and $u(s)$ to be antisymmetric. We can therefore integrate between the limits $(0, \infty)$:

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\mu^2)}\int_{0}^{\infty}(v'^2+u^2)\,ds.$$

The above expression for the energy $\bar{U}$ is substantially a function of the shape of the shell in the transitive zone A_{12} which, in turn, is defined by the functions u, v prescribing the deformation. As in previous studies we shall evaluate the energy $\bar{U}$ by using the condition that it be a minimum under a given general deformation. We characterize this deformation by a deflection h with the region of arching and in the vicinity of the given point of curve γ at which energy $\bar{U}$ is considered. All this assumes a definite meaning as the width of the transitive zone A_{12} decreases indefinitely. Under our present assumption regarding the character of postcritical deformation, h is a constant defined by the displacement of the zone of bulging. In the more general study, to be undertaken later on, h varies along the curve γ.

In terms of the initial variables v and s, h can be expressed as

$$h = -\int_{-}^{\varepsilon^*} v'ds.$$

If we change to new variables and change the limits of integration $\bar{\varepsilon}^*$ and $-\bar{\varepsilon}^*$ to $\pm\infty$, we obtain

$$h = -\frac{1}{12^{1/4}}\sqrt{\delta\rho\alpha}\int_{-\infty}^{\infty} v\,ds.$$

Finally, by taking into account the expected symmetry of the function $v(s)$, we obtain

$$h = -\frac{2}{12^{1/4}}\sqrt{\delta\rho\alpha}\int_{0}^{\infty} v\,ds.$$

Thus, the energy $\bar{U}$, as well as the function u, v on which it depends, are determined from the condition that the functional

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\mu^2)}\int_{0}^{\infty}(v'^2+u^2)\,ds$$

be a minimum, given that

$$-\frac{2}{12^{1/4}}\sqrt{\delta\rho\alpha}\int_{0}^{\infty} v\,ds = h = \text{const}.$$

In addition to the above integral equation, the variable functions u, v must satisfy the equation

$$u^1 + v + \frac{v^2}{2} = 0$$

and their values at infinity must be zero.

2. Solution of the variational problem for the functional $\bar{U}$

Let us examine the problem of the minimum value of the functional U. We first transform the equation

$$-\frac{2\sqrt{\delta\rho\alpha}}{12^{1/4}}\int_0^\infty v ds = h$$

using the relation

$$u' + v + \frac{v^2}{2} = 0.$$

If we integrate this equation between the limits $-\infty$ and ∞ and use the fact that $u(-\infty) = u(\infty) = 0$, we obtain

$$-\int_{-\infty}^{\infty} v ds = \int_{-\infty}^{\infty} \frac{v^2}{2} ds.$$

Further, taking into account the symmetry of the function $v(s)$, we obtain

$$-\int_0^\infty v ds = \frac{1}{2}\int_0^\infty v^2 ds.$$

It follows that the integral equation in $v(s)$ can be presented in the form

$$\frac{\sqrt{\delta\rho\alpha}}{12^{1/4}}\int_0^\infty v^2 ds = h = \text{const}.$$

Consequently, our variational problem consists of evaluating the minimum of the functional

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\mu^2)}\int_0^\infty (v'^2 + u^2)\, ds$$

under the following conditions:

$$\frac{\sqrt{\delta\rho\alpha}}{12^{1/4}}\int_0^\infty v^2 ds = h = \text{const}.$$

$$u' + v + \frac{v^2}{2} = 0,$$

$$u(0) = u(\infty) = v(\infty) = 0.$$

Since we are only interested in the initial stage of postcritical deformation, we can omit the term $v^2/2$ in the relation

$$u' + v + \frac{v^2}{2} = 0$$

thus obtaining the simplified form

$$u' + v = 0.$$

If we substitute v everywhere instead of u' our problem reduces to finding the minimum of the functional

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\mu^2)} \int_0^\infty (u''^2 + u^2)\, ds$$

where

$$\frac{\sqrt{\delta\rho\alpha}}{12^{1/4}} \int_0^\infty u'^2\, ds = h = \text{const}$$

and the boundary conditions

$$u(0) = u(\infty) = 0$$

are satisfied.

In accordance with the Euler-Lagrange method our variational problem is reduced to the investigation of the unconditional extremum of the functional

$$W = \int_0^\infty \left\{ \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\mu^2)} (u^2 + u''^2) + \lambda \frac{\sqrt{\delta\rho\alpha}}{12^{1/4}} u'^2 \right\} ds,$$

where λ is some constant.

Assuming for the sake of brevity that

$$-\sigma = \frac{\sqrt{12}\,(1-\mu^2)\,\rho\lambda}{E\alpha^2\delta^2},$$

we may consider our problem as reduced to finding the extremum of the functional

$$I = \int_0^\infty (u^2 + u''^2 - \sigma u'^2)\, ds,$$

which differs from W by a constant factor.

The Euler-Lagrange equation for the functional I is

$$u^{\mathrm{IV}} + u + \sigma u'' = 0,$$

and its general solution is given by

$$u(s) = \sum c_k e^{\omega_k s},$$

where ω_k denote the roots of the characteristic equation:

$$\omega^4 + 1 + \sigma\omega^2 = 0.$$

It is evident that in the complex plane $z = x + iy$ these roots are placed symmetrically with respect to the origin and the x-axis and consequently their absolute value equals unity.

In order to satisfy the boundary condition $u(\infty) = 0$, there must be two roots amongst the roots ω_k with negative real parts. If we denote these two roots by ω_1 and ω_2, the solution of our variational problem is given by a function $u(s)$ of the type

$$u = c_1 e^{\omega_1 s} + c_2 e^{\omega_2 s}.$$

Further, in order that the second boundary condition $u(0) = 0$ be satisfied, we must have $c_1 = -c_2 = c$. In such a case we have the following expression for $u(s)$:

$$u = c\,(e^{\omega_1 s} - e^{\omega_2 s}).$$

Substituting the above function into the expression for the functional $\bar{U}$ and the relevant integral connection, we obtain

$$\int_0^\infty u'^2 ds = -c^2\left(\frac{\omega_1}{2} + \frac{\omega_2}{2} - \frac{2\omega_1\omega_2}{\omega_1 + \omega_2}\right),$$

$$\int_0^\infty u^2 ds = -c^2\left(\frac{1}{2\omega_1} + \frac{1}{2\omega_2} - \frac{2}{\omega_1 + \omega_2}\right),$$

$$\int_0^\infty u''^2 ds = -c^2\left(\frac{\omega_1^3}{2} + \frac{\omega_2^3}{2} - \frac{2\omega_1^2\omega_2^2}{\omega_1 + \omega_2}\right).$$

The roots ω_1 and ω_2 are complex conjugate and have absolute values of unity. We can therefore put

$$\omega_1 = e^{i\vartheta}, \quad \omega_2 = e^{-i\vartheta}.$$

Substituting these values into our integral expressions we obtain

$$\int_0^\infty u'^2 ds = c^2 \frac{\sin^2\vartheta}{\cos\vartheta},$$

$$\int_0^\infty (u^2 + u''^2)\,ds = c^2 \frac{\sin^2\vartheta}{\cos\vartheta}(2 + 4\cos^2\vartheta),$$

and hence

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\mu^2)} c^2 (2 + 4\cos^2\vartheta),$$

$$h = \frac{\sqrt{\delta\rho\alpha}}{12^{1/4}} c^2 \frac{\sin^2\vartheta}{\cos\vartheta}.$$

It follows that

$$\bar{U} = \frac{E\delta^2\alpha^2 h}{\sqrt{12}\,(1-\mu^2)\,\rho}(2 + 4\cos^2\vartheta).$$

The minimum value of $\bar{U}$, with $h = \text{const}$, is obtained for $\vartheta = \frac{\pi}{2}$ and thus we are led to the following final expression for the energy of deformation

$$\bar{U} = \frac{2E\delta^2\alpha^2 h}{\sqrt{12}\,(1-\mu^2)\,\rho}.$$

This expression for the deformation energy of the shell in the zone of substantial local bending is very important. We shall use it in our study of other, more general, cases of deformations, which are not reduceable to that of double mirror image bulging. In fact, each time that bulging of the shell resulting from loss of stability is effected throughout zone A_2, bounded by curve γ, we shall calculate the deformation energy value in regions of considerable local bending in the vicinity of γ, by the same formula

$$\bar{U} = \frac{2E\delta^2\alpha^2 h}{\sqrt{12(1-\mu^2)}\,\rho}.$$

In such a general case α will denote the angle between the plane of contact of the γ curve and the tangent planes of the surface; ρ, the radius of curvature of the curve; and h, the change in shell deflection across the boundary of the bulging zone.

In our previous investigations of postcritical deformations, we evaluated the energy of deformation by considering both the energy in the zone of considerable local bending and energy of bending along the original shell surface. In our present study of the initial stage of postcritical deformation it may seem advisable to proceed in a similar manner. However, we soon see that in our present case the deformation energy along the original shell surface is negligible and can be neglected. For the special case of shell deformation which can be approximated by double mirror image bulging this is quite evident, since there is no change in curvature under deformation in each one of the zones A_1 and A_2. Let us investigate a general case.

Firstly, the total deformation energy in the zone of strong local bending is of the order of magnitude

$$E\delta^2\alpha^2 h.$$

This follows from the fact that the energy per unit length of γ is of the order of magnitude $E\delta^2\alpha^2 h/\rho$ and the length γ is of the order ρ. Furthermore, the change in curvature of the middle surface inside the zone of bulging, [illegible]

follows, therefore, that the energy of deformation inside the bulging zone is of the order of magnitude

$$E\delta^3\left(\frac{h}{\rho^2}\right)^2\rho^2.$$

It is quite natural to assume that the energy of bending on the remaining shell surface is of the same order of magnitude.

Thus, at the initial stage of postcritical deformation the order of magnitude of energy within the zone of considerable local bending is

$$E\delta^2\alpha^2 h,$$

and that of deformation energy along the remaining surface is

$$\frac{E\delta^3 h^2}{\rho^2}.$$

Taking into account that α is of the order of magnitude ρ/R (where R is the normal curvature of the shell surface), we see that our problem is reduced to comparing the order of magnitude of two quantities

$$\frac{E\delta^2 h\rho^2}{R^2}, \quad E\frac{\delta^3 h^2}{\rho^2}.$$

Taking into account the usual relationships between the various parameters, it is evident that the second quantity is of a smaller order of magnitude.

The above discussion leads us to the conclusion that when considering the energy of postcritical deformation at its initial stage, it is sufficient to take into account the energy in the zone of strong local bending only.

3. Evaluation of the upper critical load for a shell of positive Gaussian curvature subjected to a uniform external pressure

We define a load p acting on a shell to be a critical load if there exists the possibility that under the action of such a load the shell, in addition to its basic deflected shape of elastic stability, might assume other shapes, very close to the basic shape, and accompanied by bulging. The least value of such a load is called upper critical load. At the moment of loss of stability by the shell, the load it carries is stationary with respect to deformations accompanied by bulging. Making use of the equilibrium condition of the shell under such circumstances, the above fact enables us to evaluate, with some approximation, the value of the upper critical load. Our method can then be applied when studying such elastic shell conditions.

Elastic equilibrium of the shell is characterized by the fact that the functional

$$W = U - A,$$

must be constant. Here, U denotes the energy of elastic deformation of the shell and A the work done by the imposed loads. In the case under consideration the shell is shallow, rigidly fixed at its edges, and of positive Gaussian curvature. For such a shell the deformation energy under bulging is mainly concentrated along the boundary of bulging and its value, per unit length of curve γ defining the zone of bulging, is given by the formula

$$\bar{U} = \frac{2E\delta^2\alpha^2 h}{\sqrt{12}(1-\mu^2)\rho}.$$

In the above expression α denotes the angle between the plane of contact to the curve γ and the planes tangent to the surface; ρ, the radius of curvature of γ; h, the deflection within the bulging zone; and δ, the thickness of the shell.

In our study we have approximated the arched shell shape by a double mirror image bulging, γ being a plane curve. When the bulging zone is small and the shell surface sufficiently regular, this curve approximates closely to an ellipse similar to the indicatrix of curvature.

Let us introduce a system of rectilinear coordinates, taking the center of bulging as the origin, the tangent plane as the xy plane, and directing the x-and y-axes along the lines of curvature of the surface. The bulging zone boundary γ can then be defined by the equation

$$x = \lambda\sqrt{R_1}\cos t, \quad y = \lambda\sqrt{R_2}\sin t,$$

where R_1 and R_2 are the main radii of curvature at the center of the bulging zone, and λ is a parameter characterizing the zone dimensions.

Let us evaluate the quantities α and ρ entering the formula of deformation energy $\bar{U}$. We have

$$\frac{1}{\rho}=\frac{\sqrt{R_1R_2}}{\lambda(R_1\sin^2 t+R_2\cos^2 t)^{3/2}}.$$

In accordance with the formula of Menier

$$\alpha\simeq\rho k_n,$$

where k_n denotes the normal curvature of the shell surface in the direction of the tangent to the curve γ. Following Euler's formula

$$k_n=\frac{1}{R_1}\left(\frac{R_1\sin^2 t}{R_1\sin^2 t+R_2\cos^2 t}\right)+\frac{1}{R_2}\left(\frac{R_2\cos^2 t}{R_1\sin^2 t+R_2\cos^2 t}\right),$$

$$k_n=\frac{1}{R_1\sin^2 t+R_2\cos^2 t}.$$

An element of arc of the curve γ is given by

$$ds=\lambda(R_1\sin^2 t+R_2\cos^2 t)^{1/2}dt.$$

Substituting the above values in the expression for $\bar{U}$ and integrating along the arc of the curve γ, we find the full deformation energy to be

$$U=\int_\gamma\bar{U}\,ds=\int_0^{2\pi}\frac{2E\delta^2h\lambda^2\,dt}{\sqrt{12}(1-\mu^2)\sqrt{R_1R_2}},$$

$$U=\frac{4\pi E\delta^2h\lambda^2}{\sqrt{12}(1-\mu^2)\sqrt{R_1R_2}}.$$

Let us now calculate the work done by the external load. We have

$$A=Qh.$$

where Q denotes the total load acting on the bulging surface and h, the deflection. Further,

$$Q=pS,$$

where S is the bulging zone surface and p the pressure. In the case under consideration, S, being the area of an ellipse with semiaxes equal to $\lambda\sqrt{R_1}$ and $\lambda\sqrt{R_2}$ is equal to

$$\pi\lambda^2\sqrt{R_1R_2}.$$

Hence

$$A=Qh=pSh=\pi p\sqrt{R_1R_2}h\lambda^2.$$

Now, making use of the equilibrium condition

$$d(U-A)=0,$$

we are in a position to calculate the load supported by the shell. We have,

$$d\left\{\frac{4\pi E\delta^2h\lambda^2}{\sqrt{12}(1-\mu^2)\sqrt{R_1R_2}}-\pi p\sqrt{R_1R_2}h\lambda^2\right\}=0,$$

and it follows that

$$p=\frac{2E}{\sqrt{3}(1-\mu^2)}\frac{\delta^2}{\sqrt{R_1R_2}}.$$

As was to be expected, load p is stationary with respect to the parameter $h\lambda^2$ characterizing bulging.

Finally, the upper critical pressure for a shallow shell of positive Gaussian curvature rigidly fixed at its edges is given by the formula

$$p_e = \frac{2E\delta^2}{\sqrt{3(1-\mu^2)}\,R_1R_2},$$

where R_1, R_2 are the main radii of curvature of the shell; δ, the thickness; E, the modulus of elasticity; μ, Poisson's ratio. It should be noted that $1/R_1R_2$ is the Gaussian curvature. It follows, therefore, that the above formula can be expressed in the form

$$p_e = \frac{2E\delta^2K}{\sqrt{3(1-\mu^2)}},$$

where K is the Gaussian curvature of the middle surface of the shell.

For a spherical shell of radius R,

$$R_1 = R_2 = R,$$

and hence the formula for the critical pressure is reduced to

$$p_e = \frac{2E}{\sqrt{3(1-\mu^2)}}\left(\frac{\delta}{R}\right)^2.$$

Considering that

$$1-\mu^2 \simeq 1,$$

the above formula expresses a well-known result regarding spherical shells, namely

$$p_e = \frac{2E}{\sqrt{3(1-\mu^2)}}\left(\frac{\delta}{R}\right)^2.$$

As stated previously, we shall study the value of the upper critical pressure for shells of positive Gaussian curvature on the basis of more general assumptions regarding the mode of bulging, neglecting the case of double mirror image reflection. Such a study will, however, show that the value of critical pressure remains the same.

In conclusion we wish to make the following remark regarding the upper critical load for shells subjected to a nonuniform external pressure. The formula giving the value of the load supported by the shell under bulging shows that such a value is independent of the parameter $h\lambda^2$, which characterizes the deformation, and in particular, of the size of the bulging zone (parameter λ). We are therefore entitled to conclude that in the case of a nonuniform, but gradually changing external pressure the critical load is determined by the value of the maximal pressure.

4. Loss of stability and critical loads for various other cases of loading by external pressure

It was shown above that the result achieved regarding loss of stability of a shell of positive Gaussian curvature when loaded by external pressure is valid for cases other than that of a uniform load. To some extent this fact can be used in evaluating the critical load when the shell is loaded by a continuous, but not necessarily, uniform pressure. We will now investigate other cases of loading, namely, those where the load applied to the shell surface is not continuous. Two such cases of loading are of basic importance: a load applied along a certain line, and a concentrated load.

The case of a shell of positive Gaussian curvature subject to a concentrated load has already been studied by us /1, 2/. We showed that such a mode of loading does not cause loss of stability. It follows, therefore, that for a complete investigation of the problem we need only examine the case where the load is applied along some line on the shell surface. It is not difficult to find an example to illustrate such a case. We consider a shell subjected to the pressure of a tightly drawn thread strung across its surface and will examine the relevant questions of loss of stability and critical load.

Consider a shell of positive Gaussian curvature rigidly fixed at its edges and subjected to the pressure of a thread tightly stretched along some arc of its surface (Figure 3a). For some value of the tensile force Q, loss of stability will occur accompanied by formation of bulging zones along the line of contact (Figure 3b). We shall evaluate the value of such a critical tensile force presently.

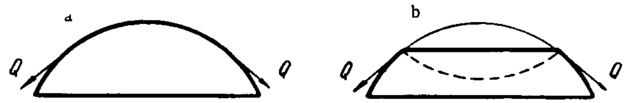

FIGURE 3.

In contrast to the case previously considered, of pressure distributed along the surface, when loss of stability is accompanied by a simultaneous bulging of some finite zone of the shell, in our present case, bulging commences to spread out from some central point situated on the line of contact between the thread and the shell.

It seems quite natural to approximate the shape of the shell under post-critical deformation to that of a simple mirror image bulging, as was done in /1, 2/. In such a case the energy of elastic deformation is given by the expression

$$U = \pi c E (2h)^{3/2} \delta^{5/2} (k_1 + k_2),$$

where $2h$ denotes deflection at the center of bulging; k_1 and k_2 are the principal curvatures of the shell; δ is the thickness; E, the modulus of elasticity; and c is a constant. In accordance with the latest data available

$$c = \frac{0.178}{1 - \mu^2},$$

μ being Poisson's ratio.

The work done by the tensile force is

$$A = Q \Delta l,$$

where Δl is the finite displacement of the thread ends due to shell bulging.

Assuming the absence of friction between the thread and the shell surface and that, in consequence, the arc of contact between the thread and the shell is a geodetic, we can easily determine the value of Δl. It equals the difference between the arc AB and the chord joining its ends. Denoting the normal curvature of the shell surface in the direction of the thread by k_n, and the deflection at the center of bulging by $2h$, we have

$$\Delta l = \frac{1}{3}(2h)^{3/2}\sqrt{k_n}.$$

If we now introduce a system of rectilinear coordinates x, y, and z, in such a way that the xy plane is the plane of contact of the thread at the center of bulging and the x-axis is along the tangent, then the shape of the thread, being in contact with the surface along a geodetic, will be given by the equation

$$y = \frac{k_n x^2}{2} + 0(x^3), \quad z = 0(x^3),$$

where $0(x^3)$ denotes quantities of the order of x^3.

If the length of the chord AB equals $2d$, then the length of the arc AB is given by

$$s = \int_{-d}^{d} \sqrt{1 + k_n^2 x^2}\, dx \simeq \int_{-d}^{d} \left(1 + \frac{k_n^2 x^2}{2}\right) dx = 2d + \frac{k_n^2 d^3}{3}.$$

Noting that $2h \simeq k_n d^2$, the above expression reduces to

$$s = 2d + \frac{1}{3}(2h)^{3/2}\sqrt{k_n}.$$

Hence the expression derived previously,

$$\Delta l = s - 2d = \frac{1}{3}(2h)^{3/2}\sqrt{k_n}.$$

Substitution of the above value of Δl into the expression for work A leads to

$$A = \frac{1}{3} Q\,(2h)^{3/2}\sqrt{k_n}.$$

The load supported by the shell is determined with the help of the equilibrium condition

$$d\,(U - A) = 0,$$

where the deflection $2h$ is varied. We have

$$d\left\{\pi c E\,(2h)^{3/2}\delta^{5/2}(k_1 + k_2) - \frac{1}{3} Q\,(2h)^{3/2}\sqrt{k_n}\right\} = 0,$$

and finally

$$Q = 3\pi c E \delta^{5/2}(k_1 + k_2)\frac{1}{\sqrt{k_n}}.$$

As in the case of continuous loading of the shell surface discussed previously, tension Q is stationary with respect to the parameter $2h$; this characterizes bulging.

Finally, the critical thread tension which might cause the shell to lose its stability and start to bulge is given by the formula

$$Q_e = 3\pi c E \delta^{5/2} (k_1 + k_2) \frac{1}{\sqrt{k_n}}.$$

In the particular case of a spherical shell of radius R,

$$k_1 = k_2 = \frac{1}{R},$$

and the formula for the critical tension is reduced to

$$Q_e = 6\pi c E \delta^2 \sqrt{\frac{\delta}{R}}.$$

In conclusion we shall evaluate the critical pressure acting on a shell of positive Gaussian curvature as transmitted by a plane [flat] elastic support (Figure 4). It was pointed out in paragraph 3 that where loading is done by uniform pressure, the value of the critical load p_e is not a function of the dimensions of the preassumed bulging zone. We conclude, therefore, that in the case when the shell is loaded by pressure of an elastic support, loss of stability will occur at the moment when the pressure, at some point of contact between shell and support, will reach the above-found critical value

$$p_e = \frac{2E\delta^2}{\sqrt{3(1-\mu^2)R_1R_2}}.$$

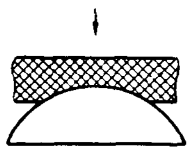

FIGURE 4.

It follows, therefore, that maximum support deflection, h, at the moment of loss of stability, is given by the relation

$$p_e = hE',$$

where E' is the rigidity of the support.

Deflection of the support at any given point is expressed by

$$z = h - \frac{1}{2}\left(\frac{1}{R_1}x^2 + \frac{1}{R_2}y^2\right),$$

where R_1 and R_2 are the principal radii of curvature of the shell at the center of contact with the support. It is not difficult to evaluate thc total force acting on the shell, namely, the required critical force Q_e.

$$Q_e = \iint F'z\,dx\,dy,$$

where integration is carried out along the surface of contact between the shell and its support.

$$Q_e = \pi E h^2 \sqrt{R_1 R_2}.$$

Substituting in the above expression the value of h as given by the relation

$$p_e = hE',$$

we obtain

$$Q_e = \pi \sqrt{R_1 R_2} \frac{p_e^2}{E'},$$

where p_e is the upper critical load under uniform external pressure.

§ 2. SPECIAL ISOMETRIC TRANSFORMATION OF A SURFACE OF POSITIVE GAUSSIAN CURVATURE

Bearing in mind the fact that we are identifying, in some definite manner, the postcritical deformation of an elastic shell with the geometric bending of its middle surface, we shall proceed in this section to the study of a special case of isometric transformations of convex surfaces. The simplest case of such a transformation is that of a double mirror reflection. Results obtained from such a study will be used to solve various problems of post-critical shell deformations at the initial stage of bulging.

1. Formulation of the bending problem and method of approach to its solution

Let F denote a regular surface of positive Gaussian curvature and let γ be a closed curve on it enclosing a region G. Further, let γ' be a curve on the above enclosed surface and G', a portion of G, enclosed by γ'. We wish to consider the problem of the isometric transformation of surface F accompanied by caving in of region G' and formation of ribs along the curves γ and γ' (Figure 5). When γ and γ' are plane curves, such a transformation is obtained by a mirror reflection of region G in the plane of curve γ, followed by a subsequent mirror reflection of a part of it, G', in the plane of curve γ'.

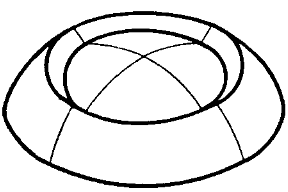

FIGURE 5.

Taking into account future applications, the problem under consideration is of interest to us for the special case when curve γ' is close to γ and region G, bounded by curve γ, is small and of elliptic shape. It is under such simplifying assumptions that we shall study our problem, and the method of solution offered is described below.

In the special case when curves γ and γ' coincide, the solution is trivial and the isometric surface transformation corresponds to the original surface F. It is natural to assume that when curves γ and γ' are close enough, the isometric transformation of surface F is different with regard to substantial deformations only within the band enclosed by these two curves. As far as the rest of the surface is concerned, finite bendings may be replaced by infinitely small ones.

We shall not proceed to explore the structure of the transformed surface within the band between curves γ and γ', since the deformation of this region of an elastic shell will be determined by considerations of energetics. We shall characterize bending of the above-mentioned band by some general relation which will enable us to determine the conjugation conditions of infinitely small bendings, outside region G and inside region G', in the limiting case

$$\gamma' \to \gamma.$$

Let A be any point on the curve γ. Consider a geodetic perpendicular from this point directed inward into region G up to its intersection with curve γ' at point A'. Denote the length of this perpendicular by δ. Under transition from surface F to the isometrically transformed surface, points A and A' will be displaced by τ_A and $\tau'_{A'}$ respectively, where τ and τ' denote the bending fields of surface F at the respective regions. We shall determine the value of the expression $\tau_A - \tau'_{A'}$ assuming curves γ and γ' to be close enough.

Formation of a rib along curve γ, under an isometric transformation of surface F, is accompanied by a rotation of the tangent plane with respect to the tangent to curve γ. In passing to the limit $\gamma' \to \gamma$ such a rotation is reduced to a mirror reflection in the plane of contact to curve γ. It follows, therefore, that, when curves γ, γ' are close enough, we may consider vector [illegible] γ, i.e., directed along the binormal to this curve:

$$\tau_A - \tau'_{A'} = \sigma e,$$

where e denotes the unit binormal vector. As far as the multiplier σ is concerned, whenever angle α between the plane of contact to curve γ and the tangent planes to the surface is small, its value is equal to $2\alpha\delta$.

As previously, because of forthcoming applications we are interested in the special case when curves γ and γ' are close to each other. Under such an assumption we pass to the limit $\gamma' \to \gamma$. The problem of the bending of surface F is thus reduced to that of finding fields of infinitesimally small bendings τ' within region G, and τ outside this region, such that along the common boundary, γ, of the region they satisfy the condition

$$\tau - \tau' = \sigma e.$$

In the above expression e denotes the unit vector along the binormal to the curve γ, and σ is some function defined along this curve.

In order to simplify the presentation of the subject, we shall consider first the problem of finding fields τ and τ' in the case when the elliptic region G is coaxial with the indicatrix of curvature of the surface at a given point. We shall call this the simple case. Later on we shall consider the general case, namely, when the assumption of coaxiality is not valid.

2. General representation for bending fields τ and τ'

Let P be the center of the caved-in region G. Taking into consideration the fact that any substantial deformations of the surface F are confined to the immediate neighborhood of point P, it is natural to introduce a rectangular coordinate system, xyz, with its origin at P and such that the tangent plane at P be the xy plane, and the normal to the surface coincide with the z-axis. Further, if the x-, y-axes be directed along the principal directions at point P, then the surface in the vicinity of P can be described by the equation

$$z = \frac{1}{2}(ax^2 + by^2),$$

where a and b denote the principal curvatures of the surface at P. In the simplest case, when region G is coaxial with the indicatrix of curvature at P, it can be defined by the inequality

$$Ax^2 + By^2 \leqslant 1.$$

Let us introduce new coordinates u, v on the surface, defined by

$$u = x\sqrt{a}, \quad v = y\sqrt{b}.$$

In these new coordinates our surface is described by the equations

$$x = \frac{u}{\sqrt{a}}, \quad y = \frac{v}{\sqrt{b}}, \quad z = \frac{1}{2}(u^2 + v^2).$$

Let ξ, η, ζ be the components of the bending field along the axes x, y, z respectively. From the equation of infinitely small bendings

$$dr\, d\tau = 0,$$

where r is a surface point vector, and τ, a bending field vector, we obtain the following system of equations for the functions ξ, η, ζ:

$$\frac{1}{\sqrt{a}}\xi_u + u\zeta_u = 0,$$
$$\frac{1}{\sqrt{b}}\eta_v + v\zeta_v = 0,$$
$$\frac{1}{\sqrt{b}}\eta_u + \frac{1}{\sqrt{a}}\xi_v + u\zeta_v + v\zeta_u = 0.$$

Upon elimination of functions ξ and η from the above equations, we obtain the Laplace equation for ζ,

$$\frac{d^2\zeta}{du^2} + \frac{d^2\zeta}{dv^2} = 0.$$

Introducing the expression

$$w = u + iv,$$

we can describe the general representation for the ζ component with the help of the analytic function of the complex variable w as follows:

$$\zeta = \mathrm{Re}\,\zeta(w).$$

The remaining two components, ξ and η of the bending field can now be described with the help of the function $\zeta(w)$, by means of the formulas

$$\xi = \sqrt{a}\,\mathrm{Re}\left(-u\zeta + \int \zeta\, dw\right),$$
$$\eta = \sqrt{b}\,\mathrm{Re}\left(-v\zeta - i\int \zeta\, dw\right).$$

The representation for bending fields derived above, applies equally well to the general case. However, bearing in mind the solution of the conjugation problem, it will be more convenient to give a somewhat different form to the bending field representation in the general case.

The caved-in region G can, in the general case, be described with the help of the inequality

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \leqslant 1.$$

Let us introduce new variables u, v defined by the formulas

$$x = \lambda_{11}u + \lambda_{12}v,$$
$$y = \lambda_{21}u + \lambda_{22}v.$$

We shall select the coefficients λ_{ij} in such a manner that, in the new coordinates u, v the surface shall be given by the equation

$$z = \frac{1}{2}(u^2 + v^2),$$

and the caved-in region G, by the inequality

$$Au^2 + Bv^2 \leqslant 1.$$

The possibility of such a selection of the coefficients λ_{ij} is made certain by the positive definiteness of the quadratic forms

$$\frac{1}{2}(ax^2 + by^2) \quad \text{[illegible]}$$

which, with the help of the above-mentioned transformation, are reduced simultaneously to the canonical form.

Further, let us introduce the variables $\bar{x}$, $\bar{y}$ defined by the equations

$$\bar{x} = x\sqrt{a}, \quad \bar{y} = y\sqrt{b}.$$

This transformation again reduces the expression for z to a sum of squares

$$z = \frac{1}{2}(\bar{x}^2 + \bar{y}^2).$$

It is obvious that the transformation of variables $\bar{x}$, $\bar{y}$ into u, v is orthogonal and is given by the formulas

$$\bar{x} = u\cos\vartheta - v\sin\vartheta,$$
$$\bar{y} = u\sin\vartheta + v\cos\vartheta.$$

It follows, therefore, that the mutual interdependence between the variables x, y and u, v is defined by the expressions

$$x = \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),$$
$$y = \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta).$$

Angle ϑ is evaluated by making use of the condition that our transformation reduces the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

into

$$Au^2 + Bv^2.$$

Quantities A and B are then eigenvalues of the expression

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

with respect to

$$ax^2 + by^2$$

and are, therefore, the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - \lambda a, & a_{12} \\ a_{21}, & a_{22} - \lambda b \end{vmatrix} = 0.$$

It was shown above that the bending field, when expressed through the variables $\bar{x}$, $\bar{y}$, is defined by the equations

$$\zeta = \operatorname{Re} \bar{\zeta}(\bar{z}),$$
$$\xi = \sqrt{a} \operatorname{Re}(-\bar{x}\bar{\zeta} + \int \bar{\zeta}\, d\bar{z}),$$
$$\eta = \sqrt{b} \operatorname{Re}(-\bar{y}\bar{\zeta} - i \int \bar{\zeta}\, d\bar{z}),$$

where $\bar{\zeta}(\bar{z})$ is an analytic function of the complex variable $\bar{z} = \bar{x} + i\bar{y}$.

Let us rewrite the above expressions in terms of the variables u, v. Bearing in mind that

$$\bar{z} = we^{i\vartheta}, \quad w = u + iv,$$

we have

$$\zeta = \operatorname{Re} \zeta(w),$$
$$\xi = \sqrt{a} \operatorname{Re}(-(u \cos\vartheta - v \sin\vartheta)\,\zeta + e^{i\vartheta} \int \zeta\, dw),$$
$$\eta = \sqrt{b} \operatorname{Re}(-(u \sin\vartheta + v \cos\vartheta)\,\xi - ie^{i\vartheta} \int \zeta\, dw),$$

where $\zeta(w) = \bar{\zeta}(we^{i\vartheta})$ is an analytic function of the complex variable w.

Such is the representation for surface bending fields in the most general case.

3. Conjugation of bending fields τ and τ' in the simplest case

The problem of isometric transformation of a convex surface F, presented in paragraph 1, was reduced, in paragraph 2, to that of constructing two fields of infinitesimally small bendings, τ –beyond region G and τ'–inside G, such that on their mutual boundary, γ, they satisfy the conjugation condition

$$\tau - \tau' = \sigma e.$$

In the above formula e denotes the unit binormal vector of the curve γ, and

σ is some given function on the curve. Presently, we shall complete the solution of the bending problem by constructing the bending fields τ and τ'.

In paragraph 2 we found a general representation of the bending fields by making use of the analytic function of the complex variable $w = u + iv$, namely

$$\zeta = \operatorname{Re} \zeta(w),$$
$$\xi = \sqrt{a}\operatorname{Re}(-u\zeta + \int \zeta\, dw),$$
$$\eta = \sqrt{b}\operatorname{Re}(-v\zeta - i\int \zeta\, dw).$$

Analytic function $\zeta'(w)$, within region G, i.e., inside the ellipse

$$\frac{A}{a}u^2 + \frac{B}{b}v^2 \leqslant 1,$$

corresponds to the bending field τ', while the analytic function $\zeta(w)$, outside the ellipse

$$\frac{A}{a}u^2 + \frac{B}{b}v^2 \geqslant 1,$$

corresponds to the bending field τ. The difference between the bending fields along the curve γ, i.e., under

$$\frac{A}{a}u^2 + \frac{B}{b}v^2 = 1,$$

which is of interest to us, is given by the system of equations

$$\Delta\zeta = \operatorname{Re} \Delta\zeta(w)$$
$$\Delta\xi = \sqrt{a}\operatorname{Re}(-u\Delta\zeta + \int \Delta\zeta\, dw),$$
$$\Delta\eta = \sqrt{b}\operatorname{Re}(-v\Delta\zeta - i\int \Delta\zeta\, dw),$$

where $\Delta\zeta(w)$ denotes the difference between the analytic functions $\zeta(w)$ and $\zeta'(w)$ on the ellipse

$$\frac{A}{a}u^2 + \frac{B}{b}v^2 = 1.$$

We shall now pass from the complex variable w to the variable ω, assuming that

$$w = \lambda\omega + \frac{\mu}{\omega}.$$

Let us select the constants λ and μ in such a manner that to the circle $|\omega| = 1$ on the plane of complex variable ω, there should correspond the ellipse

$$\frac{A}{a}u^2 + \frac{B}{b}v^2 = 1$$

on the plane w. Evidently, this can be achieved by submitting the quantities λ, μ to the requirements

$$\lambda + \mu = \sqrt{\frac{a}{A}}, \quad \lambda - \mu = \sqrt{\frac{b}{B}}.$$

On the boundary of the G region, i. e., on the ellipse

$$\frac{A}{a}u^2+\frac{B}{b}v^2=1,$$

$\omega=e^{i\varphi}$, and therefore

$$u=(\lambda+\mu)\cos\varphi,\quad v=(\lambda-\mu)\sin\varphi.$$

In the formulas defining $\Delta\tau=\tau-\tau'$ we shall effect the transition from variable w to $\omega=e^{i\varphi}$, by assuming

$$\Delta\zeta(w)=P(\varphi)+iQ(\varphi).$$

We thus obtain

$$\begin{gathered}\Delta\zeta=P,\\ \Delta\xi=-\sqrt{a}\{(\lambda+\mu)\cos\varphi P+\int((\lambda+\mu)\sin\varphi P+\\ +(\lambda-\mu)\cos\varphi Q)\,d\varphi\},\\ \Delta\eta=\sqrt{b}\{-(\lambda-\mu)\sin\varphi P+\int((\lambda-\mu)\cos\varphi P-\\ -(\lambda+\mu)\sin\varphi Q d\varphi)\}.\end{gathered}$$

Let us write down the formula of the curve γ (the boundary of the G region on the surface) by taking the angle $\varphi=\arg\omega$ as parameter on the curve. We have

$$\begin{gathered}x=\frac{u}{\sqrt{a}}=\frac{\lambda+\mu}{\sqrt{a}}\cos\varphi,\\ y=\frac{v}{\sqrt{b}}=\frac{\lambda-\mu}{\sqrt{b}}\sin\varphi,\\ z=\frac{u^2+v^2}{2}=\frac{\lambda^2+\mu^2}{2}+\lambda\mu\cos 2\varphi,\end{gathered}$$

and, therefore, the curve is defined by the equations

$$\begin{gathered}x=\frac{\lambda+\mu}{\sqrt{a}}\cos\varphi,\ y=\frac{\lambda-\mu}{\sqrt{b}}\sin\varphi,\\ z=\frac{\lambda^2+\mu^2}{2}+\lambda\mu\cos 2\varphi.\end{gathered}$$

Let us evaluate the vector of the binormal to curve γ. Its components along the x-, y-, and z-axes are the minors of the matrix

$$\begin{pmatrix}x', & y', & z'\\ x'', & y'', & z''\end{pmatrix}.$$

We use the notation

$$a_1=\begin{vmatrix}y'z'\\ y''z''\end{vmatrix},\ a_2=\begin{vmatrix}z'x'\\ z''x''\end{vmatrix},\ a_3=\begin{vmatrix}x'y'\\ x''y''\end{vmatrix}.$$

Omitting all the intermediate computations, we give below the final expressions for a_1, a_2, and a_3:

$$\begin{gathered}a_1=-\frac{4}{\sqrt{b}}(\lambda-\mu)\lambda\mu\cos^3\varphi,\\ a_2=\frac{4}{\sqrt{a}}(\lambda+\mu)\lambda\mu\sin^3\varphi,\\ a_3=\frac{\lambda^2-\mu^2}{\sqrt{ab}}.\end{gathered}$$

The conjugation conditions for the bending fields may be expressed as follows:

$$\Delta\xi = \sigma a_1,$$
$$\Delta\eta = \sigma a_2,$$
$$\Delta\zeta = \sigma a_3.$$

The above expressions include a factor normalizing the binormal vector in σ.

Let us differentiate the conjugation conditions with respect to φ and substitute in the expressions thus obtained the above values of $\Delta\xi$, $\Delta\eta$, and $\Delta\zeta$. We obtain

$$(\Delta\zeta)' = P',$$
$$(\Delta\xi)' = -\sqrt{a}\{(\lambda+\mu)\cos\varphi\, P' + Q(\lambda-\mu)\cos\varphi\},$$
$$(\Delta\eta)' = -\sqrt{b}\{(\lambda-\mu)\sin\varphi\, P' + Q(\lambda+\mu)\sin\varphi\}.$$

The conjugation conditions can now be rewritten in the following form:

$$(\lambda+\mu)\cos\varphi\, P' + Q(\lambda-\mu)\cos\varphi = \frac{4}{\sqrt{ab}}(\lambda-\mu)\lambda\mu(\sigma\cos^3\varphi)',$$
$$(\lambda-\mu)\sin\varphi\, P' + Q(\lambda+\mu)\sin\varphi = -\frac{4}{\sqrt{ab}}(\lambda+\mu)\lambda\mu(\sigma\sin^3\varphi)',$$
$$P' = \frac{1}{\sqrt{ab}}(\lambda^2-\mu^2)\sigma',$$

or, by incorporating the factor $\frac{1}{\sqrt{ab}}$ in σ

$$(\lambda+\mu)\cos\varphi\, P' + Q(\lambda-\mu)\cos\varphi = 4(\lambda-\mu)\lambda\mu(\sigma\cos^3\varphi)',$$
$$(\lambda-\mu)\sin\varphi\, P' + Q(\lambda+\mu)\sin\varphi = -4(\lambda+\mu)\lambda\mu(\sigma\sin^3\varphi)',$$
$$P' = (\lambda^2-\mu^2)\sigma'.$$

At first glance the above conditions may lead us to suspect some in-correctness [illegible] Because of the arbitrary character of curve γ', function σ is, in reality, an arbitrary function and we thus have 3 equations for the 2 functions P and Q. However, it is easy to see that the third equation follows from the first two, and consequently we have two equations for the two functions P and Q, which we can solve.

For our purposes we have assumed a rather simple shape for the caved-in region G, namely that of an ellipse. It would seem expedient therefore, to assume for σ the simplest possible function — a constant. All further deliberations will be carried out for just such a case.

Let

$$\sigma = \text{const.}$$

Then

$$P = (\lambda^2-\mu^2)\sigma, \quad Q = -6\lambda\mu\sigma\sin 2\varphi,$$

and therefore the following condition,

$$\zeta - \zeta' = (\lambda^2-\mu^2)\sigma - 6\lambda\mu\sigma\sin 2\varphi i,$$

holds for the analytic functions ζ and ζ' defining our bending fields on the curve γ.

Let us now evaluate the analytic functions ζ and ζ' proper. We shall expect function ζ to be such that bending field τ, outside region G, defined by it will vanish at infinity. The necessity for such a stipulation is dictated by future applications. This condition will be satisfied if we require that function ζ decreases as $1/\omega^2$ at infinity.

Leaving aside the question of the single-valuedness of the solution to our problem (it appears to be unique), we shall try to look for it by assuming

$$\zeta = \frac{\alpha}{\omega^2}, \quad \zeta' = \beta\omega^2 + c,$$

where α, β, and c are some constants.

On the boundary of the G region, i.e., when $|\omega| = 1$,

$$\zeta = \frac{\alpha}{\omega^2} = \alpha(\cos 2\varphi - i\sin 2\varphi),$$

$$\zeta' = \beta\left(\lambda\omega + \frac{\mu}{\omega}\right)^2 + c = \beta\{(\lambda^2+\mu^2)\cos 2\varphi + \\ + i(\lambda^2-\mu^2)\sin 2\varphi + 2\lambda\mu\} + c.$$

Therefore,

$$\zeta - \zeta' = (\alpha - \beta(\lambda^2+\mu^2))\cos 2\varphi - \\ - i(\alpha + \beta(\lambda^2-\mu^2))\sin 2\varphi - 2\lambda\mu\beta - c.$$

Remembering now that

$$\zeta - \zeta' = (\lambda^2-\mu^2)\sigma - 6\lambda\mu\sigma\sin 2\varphi\, i,$$

we obtain the following system of equations for the constants α, β, and c:

$$\alpha - \beta(\lambda^2+\mu^2) = 0,$$
$$\alpha + \beta(\lambda^2-\mu^2) = 6\lambda\mu\sigma,$$
$$-2\lambda\mu\beta - c = (\lambda^2-\mu^2)\sigma.$$

Solving, we find

$$\beta = \frac{3\mu\sigma}{\lambda}, \quad \alpha = \frac{3\mu\sigma}{\lambda}(\lambda^2+\mu^2).$$

Once functions ζ and ζ' are evaluated we know the bending fields τ and τ' and have thus solved the bending problem of the surface F posed in section 1. In the case when the original surface is defined by means of a vector function r, the isometrically transformed surface is given by the vector function $r+\tau'$ within the caved-in region and by the vector function $r+\tau$ beyond it. Vector functions τ and τ' are evaluated with the help of the analytic functions ζ and ζ' in accordance with the formulas derived in section 2.

4. Conjugation of bending fields — general case

Just as in the simple case, discussed above, the bending fields τ and τ' on the boundary, γ, of the caved-in region satisfy the conjugation condition

$$\tau - \tau' = \sigma e,$$

where e denotes the unit binormal vector of the curve γ, and σ is some function defined on this curve.

Bending fields τ and τ' can be represented with the help of the respective functions $\zeta(w)$:

$$\zeta = \operatorname{Re}\zeta(w),$$
$$\xi = \sqrt{a}\operatorname{Re}\{-(u\cos\vartheta - v\sin\vartheta)\zeta + e^{i\vartheta}\int\zeta dw\},$$
$$\eta = \sqrt{b}\operatorname{Re}\{-(u\sin\vartheta + v\cos\vartheta)\zeta - ie^{i\vartheta}\int\zeta dw\}.$$

For the field $\tau, \zeta(w)$ is the analytic function within the region

$$Au^2 + Bv^2 \geqslant 1,$$

and the respective function, $\zeta'(w)$, for the field τ' is analytic within the region

$$Au^2 + Bv^2 \leqslant 1.$$

The boundary of these regions is an ellipse defined by the equation

$$Au^2 + Bv^2 = 1.$$

The difference between the bending fields $\tau - \tau'$, along the curve γ is given by the system of equations,

$$\Delta\zeta = \operatorname{Re}\Delta\zeta(w),$$
$$\Delta\xi = \sqrt{a}\operatorname{Re}\{-(u\cos\vartheta - v\sin\vartheta)\Delta\zeta + e^{i\vartheta}\int\Delta\zeta dw\},$$
$$\Delta\eta = \sqrt{b}\operatorname{Re}\{-(u\sin\vartheta + v\cos\vartheta)\Delta\zeta - ie^{i\vartheta}\int\Delta\zeta dw\},$$

where $\Delta\zeta(w)$ is the difference between the analytic functions $\zeta(w)$ and $\zeta'(w)$ on the ellipse

$$Au^2 + Bv^2 = 1.$$

As in the simple case let us introduce the complex variable ω assuming

$$w = \lambda\omega + \frac{\mu}{\omega};$$

the constants λ, are to be evaluated from the condition that to the unit circle $|\omega| = 1$ on the plane w there should correspond the ellipse

$$Au^2 + Bv^2 = 1.$$

For this purpose we require that

$$\lambda + \mu = \frac{1}{\sqrt{A}}, \quad \lambda - \mu = \frac{1}{\sqrt{B}}$$

On the G-region boundary, i.e., along the curve γ,

$$\omega = e^{i\varphi},$$

and, therefore,

$$u = (\lambda + \mu)\cos\varphi, \quad v = (\lambda - \mu)\sin\varphi.$$

Let us rewrite the formulas defining the difference between the bending fields, $\Delta\tau$, along the curve γ by introducing $\omega = e^{i\varphi}$ in place of the variable w. Assuming, as in the simple case, that along γ

$$\Delta\zeta(w) = P(\varphi) + iQ(\varphi),$$

we shall have

$$\begin{aligned}\Delta\xi = \sqrt{a}\,\mathrm{Re}\{-(\lambda\cos(\varphi+\vartheta)+\mu\cos(\vartheta-\varphi))(P+iQ)+\\ + ie^{i\varphi}\int(P+iQ)(\lambda e^{i\varphi}-\mu e^{-i\varphi})\,d\varphi\} =\\ = \sqrt{a}\{-(\lambda\cos(\varphi+\vartheta)+\mu\cos(\vartheta-\varphi)\}P-\\ -\int(\lambda\sin(\varphi+\vartheta)-\mu\sin(\vartheta-\varphi))\,Pd\varphi-\\ -\int(\lambda\cos(\varphi+\vartheta)-\mu\cos(\vartheta-\varphi))\,Qd\varphi\}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{d\varphi}(\Delta\xi) = -\sqrt{a}\{(\lambda\cos(\varphi+\vartheta)+\mu\cos(\vartheta-\varphi))P'+\\ +(\lambda\cos(\varphi+\vartheta)-\mu\cos(\vartheta-\varphi)Q\}.\end{aligned}$$

In a similar way we obtain,

$$\begin{aligned}\frac{u}{d\varphi}(\Delta\eta) = -\sqrt{b}\{(\lambda\sin(\vartheta+\varphi)+\mu\sin(\vartheta-\varphi)P'+\\ +(\lambda\sin(\varphi+\vartheta)-\mu\sin(\vartheta-\varphi)Q\}.\end{aligned}$$

Finally

$$\frac{d}{d\varphi}(\Delta\zeta) = P'.$$

If we denote the curve binormal vector components by a_1, a_2, and a_3, then the conjugation conditions, after differentiation with respect to φ, will be given by

$$\begin{aligned}\frac{d}{d\varphi}(\Delta\xi) &= (a_1\sigma)',\\ \frac{d}{d\varphi}(\Delta\eta) &= (a_2\sigma)',\\ \frac{d}{d\varphi}(\Delta\zeta) &= (a_3\sigma)'.\end{aligned}$$

Let us evaluate the expressions for the components a_1, a_2, and a_3 and substitute them in the above formulas.

The surface F is given in terms of the coordinates u, v by the equations

$$\begin{aligned}x &= \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),\\ y &= \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta),\\ z &= \frac{1}{2}(u^2+v^2).\end{aligned}$$

Along the curve γ

$$u = (\lambda+\mu)\cos\varphi, \quad v = (\lambda-\mu)\sin\varphi.$$

Substitution of the above values of u, v in the surface equations leads us to the equations of the curve γ

$$\begin{aligned}x &= \frac{1}{\sqrt{a}}[(\lambda+\mu)\cos\varphi\cos\vartheta - (\lambda-\mu)\sin\varphi\sin\vartheta],\\ y &= \frac{1}{\sqrt{b}}[(\lambda+\mu)\cos\varphi\sin\vartheta + (\lambda-\mu)\sin\varphi\cos\vartheta],\\ z &= \frac{\lambda^2+\mu^2}{2} + \lambda\mu\cos 2\varphi.\end{aligned}$$

Using the curve equation we evaluate the binormal vector components

$$a_1 = \begin{vmatrix} y'z' \\ y''z'' \end{vmatrix} = -\frac{4\lambda\mu}{\sqrt{b}}(\lambda+\mu)\sin\vartheta\sin^3\varphi - \frac{4\lambda\mu}{\sqrt{b}}(\lambda-\mu)\cos\vartheta\cos^3\varphi,$$

$$a_2 = \begin{vmatrix} z'x' \\ z''x'' \end{vmatrix} = \frac{4\lambda\mu}{\sqrt{a}}(\lambda+\mu)\cos\vartheta\sin^3\varphi - \frac{4\lambda\mu}{\sqrt{a}}(\lambda-\mu)\sin\vartheta\cos^3\varphi,$$

$$a_3 = \begin{vmatrix} x'y' \\ x''y'' \end{vmatrix} = \frac{1}{\sqrt{ab}}(\lambda^2-\mu^2).$$

Substituting the above values of a_1, a_2, and a_3 in the conjugation condition and incorporating the factor $1/\sqrt{ab}$ in σ, we obtain

$$\begin{gathered}
-[\lambda\cos(\varphi+\vartheta)+\mu\cos(\vartheta-\varphi)]P' - Q[\lambda\cos(\varphi+\vartheta)-\mu\cos(\vartheta-\varphi)] = \\
= -4\lambda\mu[\sigma(\lambda+\mu)\sin\vartheta\sin^3\varphi + \sigma(\lambda-\mu)\cos\vartheta\cos^3\varphi]', \\
-[\lambda\sin(\varphi+\vartheta)+\mu\sin(\vartheta-\varphi)]P' - Q[\lambda\sin(\varphi+\vartheta) - \\
-\mu\sin(\vartheta-\varphi)] = 4\lambda\mu[\sigma(\lambda+\mu)\cos\vartheta\sin^3\varphi - \sigma(\lambda-\mu)\sin\vartheta\cos^3\varphi]', \\
P' = [\sigma(\lambda^2-\mu^2)]'.
\end{gathered}$$

Multiplying the second equation by i and adding to the first yields

$$\begin{gathered}
-P'(\lambda e^{i(\varphi+\vartheta)} + \mu e^{i(\vartheta-\varphi)}) - Q(\lambda e^{i(\varphi+\vartheta)} - \mu e^{i(\vartheta-\varphi)}) = \\
= [4\lambda\mu\sigma(\lambda+\mu)\sin^3\varphi e^{i\vartheta}i - 4\lambda\mu\sigma(\lambda-\mu)e^{i\vartheta}\cos^3\varphi]'.
\end{gathered}$$

Cancelling $e^{i\vartheta}$ and separating the real and imaginary parts, we obtain

$$P'(\lambda+\mu)\cos\varphi + Q(\lambda-\mu)\cos\varphi = (4\lambda\mu(\lambda-\mu)\sigma\cos^3\varphi)',$$

In the general case, we obtain for the functions P and Q a system of equations identical to that of the simplest case. Proceeding as before, we assume

$$\sigma = \text{const}.$$

Then

$$\begin{gathered}
P = (\lambda^2-\mu^2)\sigma, \\
Q = -6\lambda\mu\sigma\sin 2\varphi.
\end{gathered}$$

Then, as in the case previously discussed, we evaluate the analytic functions ζ and ζ' which define the bending fields:

$$\begin{gathered}
\zeta = \frac{\alpha}{\omega^2}, \quad \zeta' = \beta w^2 + c. \\
\beta = -\frac{3\mu}{\lambda}\sigma, \quad \alpha = -\frac{3\mu}{\lambda}(\lambda^2+\mu^2)\sigma.
\end{gathered}$$

With the help of the functions ζ and ζ', and using the respective formulas, we determine the bending fields τ and τ', as well as the vector function which defines the isometrically transformed surface.

§ 3. LOSS OF STABILITY OF SHELLS OF ROTATION UNDER VARIOUS MODES OF LOADING

It is very probable that of the shells of positive Gaussian curvature, those of rotation are the ones that are mostly in use. Thus the study of loss of stability of this class of shells is very important. In the present section we shall study loss of stability of shells of rotation of positive Gaussian curvature when subjected to various modes of loading: internal pressure, external pressure, and torsion. In particular, we shall evaluate the critical load in each one of the above cases.

As a prerequisite to the study of loss of stability of shells of rotation we shall study the loss of stability of a shell subjected to external pressure. We studied this problem in section 1 where we considered shallow shells of positive Gaussian curvature with edges rigidly fixed. In our present study we shall start with more general assumptions regarding the character of bulging, having no connection with the case of double mirror reflection.

1. Loss of stability of a shell of positive Gaussian curvature subjected to uniform external pressure

We start with the assumption that loss of stability of a shell subjected to external pressure is accompanied by bulging of a small, but finite, region G having the shape of an ellipse. It is assumed that region G is coaxial with the indicatrix of curvature at its center P. In particular, region G and the indicatrix may be similar and placed similarly, corresponding thus to the case discussed in section 1.

We shall approximate the shape of the shell under noticeable bulging to the isometric transformation of the original surface studied in section 2. In such a case, as was shown in section 1, energy of shell deformation is concentrated, in the main, along the boundary of the bulging zone (we shall denote it by γ). For unit length of curve γ, its value is given by

$$\bar{U} = \frac{2E\delta^2\alpha^2 h}{\sqrt{12(1-\nu^2)}\,\rho}.$$

In this expression, h is the normal deflection at the zone of bulging along the boundary γ; ρ is the radius of curvature of γ; α, the angle between the plane of contact to γ and the surface tangent planes; E, the modulus of elasticity; and ν, Poisson's ratio.

For the isometric transformation of the surface constructed in section 2, and along the boundary γ of the bulging region,

$$h = \operatorname{Re}(\Delta\zeta)_\gamma,$$

where $(\Delta\zeta)_\gamma$ is the difference between the analytic functions along γ, defining the bending fields without and within region G.

Since G, the region of protrusion, is small, we can evaluate the angle α with the help of the formula

$$\alpha = \frac{k_n}{k},$$

where k is the curvature of curve γ, and k_n is the normal curvature of the original surface in the direction of γ. Let us evaluate k and k_n.

As before, let us introduce a system of rectilinear coordinates, xyz, taking the xy plane as the tangent plane at the center of bulging P, the axis z to be normal to the surface, and the origin to coincide with point P. The axes x and y are directed along the tangents to the lines of curvature at P. In such a case the surface in the vicinity of point P is defined by the equation

$$z = \frac{1}{2}(ax^2 + by^2),$$

and the first and second quadratic forms of the surface are given by

$$\mathrm{I} = dx^2 + dy^2,$$
$$\mathrm{II} = adx^2 + bdy^2.$$

It follows, therefore, that the normal curvature of the surface is given by

$$k_n = \frac{adx^2 + bdy^2}{dx^2 + dy^2}.$$

It was shown in section 2 that curve γ is defined by the equations

$$x = p\cos\varphi, \quad y = q\sin\varphi,$$

where

$$p = \frac{\lambda + \mu}{\sqrt{a}}, \quad q = \frac{\lambda - \mu}{\sqrt{b}},$$

and λ and μ are given by the relations

$$\lambda + \mu = \sqrt{\frac{a}{A}}, \quad \lambda - \mu = \sqrt{\frac{b}{B}}.$$

The constants A and B define the region of bulging G

$$Ax^2 + by^2 \leqslant 1.$$

Since region G is small, the curvature of γ may be evaluated with the help of its projection on the xy plane, defined by the pair of equations

$$x = p\cos\varphi, \quad y = q\sin\varphi.$$

Under these circumstances we obtain the following expression for the curvature:

$$k = \frac{pq}{(p^2\sin^2\varphi + q^2\cos^2\varphi)^{3/2}}.$$

Substituting in the general expression for the normal curvature of the surface

$$dx = -p\sin\varphi d\varphi, \quad dy = q\cos\varphi d\varphi,$$

we obtain the normal curvature of the surface in the γ-direction

$$k_n = \frac{(\lambda+\mu)^2 \sin^2\varphi + (\lambda-\mu)^2 \cos^2\varphi}{p^2 \sin^2\varphi + q^2 \cos^2\varphi}.$$

Further substitution of the above values in the formula for $\overline{U}$, followed by integration along the arc of curve γ, leads to an expression giving the full energy of deformation. We have

$$\frac{\alpha^2}{\rho^2} = \left(\frac{(\lambda+\mu)^2 \sin^2\varphi + (\lambda-\mu)^2 \cos^2\varphi}{p^2 \sin^2\varphi + q^2 \cos^2\varphi}\right)^2 (p^2 \sin^2\varphi + q^2 \cos^2\varphi)^{3/2} \frac{1}{pq},$$

$$ds = (p^2 \sin^2\varphi + q^2 \cos^2\varphi)^{1/2}\, d\varphi.$$

Therefore

$$\int_\gamma \frac{\alpha^2}{\rho}\, ds = \int_0^{2\pi} \{(\lambda+\mu)^2 \sin^2\varphi + (\lambda-\mu)^2 \cos^2\varphi\}^2 \frac{d\varphi}{pq} =$$

$$= (\lambda^4 + \mu^4 + 4\lambda^2\mu^2) \frac{2\pi}{pq},$$

or, remembering that

$$p = \frac{\lambda+\mu}{\sqrt{a}}, \quad q = \frac{\lambda-\mu}{\sqrt{b}},$$

we obtain

$$\int \frac{\alpha^2}{\rho}\, ds = \frac{2\pi\sqrt{ab}}{\lambda^2-\mu^2} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

The full energy of deformation is

$$U = \int_\gamma \overline{U}\, ds = \frac{2E\delta^2 h}{\sqrt{12}\,(1-\nu^2)} \frac{2\pi\sqrt{ab}}{\lambda^2-\mu^2} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

In view of the smallness of region G, and with our choice of coordinate system, we can assume h to be equal to the difference between the components of the bending fields τ and τ' along the z-axis (section 2). Then

$$h = P = (\lambda^2 - \mu^2)\,\sigma.$$

We thus obtain a final expression for shell deformation energy

$$U = \frac{2E\delta^2\sigma 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

We shall now evaluate the work, A, done by the external pressure. Let ΔV denote the change of volume, confined by the shell, under deformation. Then,

$$A = p\Delta V.$$

Since region G is small, substantial shell deformations occur in the vicinity of P. It follows, therefore, that the quantity ΔV can be evaluated with the help of the integral

$$\Delta V = \int \zeta\, dx\, dy,$$

where ζ is the displacement, under deformation, of surface points in the direction of the z-axis. The magnitude of the displacement ζ is determined with the help of the two analytic functions $\zeta(w)$ and $\zeta'(w)$ as follows: Beyond the region of bulging

$$\zeta = \operatorname{Re}\zeta(w),$$

and within the region

$$\zeta = \operatorname{Re}\zeta'(w).$$

In section 2 we obtained the expression for the functions $\zeta(w)$ and $\zeta'(w)$. We therefore have

$$\Delta V = \operatorname{Re}\left\{\iint \zeta'(w)\,dx\,dy + \iint \zeta(w)\,dx\,dy\right\}.$$

Integration of the first term is carried out throughout the inner area of the ellipse

$$Ax^2 + By^2 \leqslant 1,$$

and that of the second, throughout the remaining portion of the xy plane.

Substituting the variables u, v in place of the variables xy

$$x = \frac{u}{\sqrt{a}}, \quad y = \frac{v}{\sqrt{b}},$$

we obtain

$$\iint \zeta'(w)\,dx\,dy = \frac{1}{\sqrt{ab}}\iint \zeta'(w)\,du\,dv,$$

where integration of the right-hand side is performed throughout the inner area of the ellipse

$$\frac{A}{a}u^2 + \frac{B}{b}v^2 \leqslant 1$$

on the plane of the complex variable $w = u + iv$. For the purpose of evaluation of this integral let us examine the curvelinear integral along the boundary of the above-mentioned ellipse

$$I' = \oint \zeta'(w)\,\bar{w}\,dw.$$

Transformation of the integral I' to a surface integral along the area of the ellipse, using the Green-Ostrogradsky formula, and the fact that function $\zeta'(w)$ is analytic, leads us to

$$I' = -2i\iint \zeta'(w)\,du\,dv.$$

The integral

$$I = \oint \zeta(w)\,\bar{w}\,dw$$

is transformed in the same way as an integral taken across the exterior part of the ellipse. It should be carefully noted that at infinity $\zeta(w)$ decreases as $1/w^2$. Preserving the same sense when integrating along the ellipse boundary,

$$I = 2i \iint \zeta(w)\, du\, dv.$$

Substituting the above values of the integrals in the expression for ΔV, we obtain

$$\Delta V = \frac{1}{\sqrt{ab}} \operatorname{Re} \frac{1}{2i} \oint (\zeta(w) - \zeta'(w))\, \bar{w}\, dw.$$

It should be noted that at the integration boundary

$$\zeta(w) - \zeta'(w) = \Delta\zeta = P + iQ,$$

where P and Q have the values

$$P = (\lambda^2 - \mu^2)\,\sigma, \quad Q = -6\lambda\mu\sigma \sin 2\varphi.$$

At this stage we introduce a new variable ω where

$$w = \lambda\omega + \frac{\mu}{\omega}.$$

In the ω plane the contour of integration is the unit circle, and therefore

$$\begin{gathered} \omega = \lambda e^{i\varphi} + \mu e^{-i\varphi}, \\ \bar{\omega} = \lambda e^{-i\varphi} + \mu e^{i\varphi}, \\ \Delta\zeta = (\lambda^2 - \mu^2)\,\sigma - 3\lambda\mu\sigma\,(e^{2i\varphi} - e^{-2i\varphi}). \end{gathered}$$

After substituting the values of $\omega, \bar{\omega}$, and $\Delta\zeta$, derived above, in the integral

$$\oint_{|\omega|=1} \Delta\zeta(w)\, \bar{w}\, dw,$$

we proceed without difficulty to the integration proper and obtain the following result:

$$\oint_{|\omega|=1} \Delta\zeta(w)\, \bar{w}\, dw = 2\pi i \sigma\,(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

At the same time we obtain

$$\Delta V = \frac{\pi\sigma}{\sqrt{ab}}\,(\lambda^4 + \mu^4 + 4\lambda^2\mu^2),$$

and, therefore,

$$A = \frac{\pi p \sigma}{\sqrt{ab}}\,(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

The quantities λ and μ characterize the shape of the bulging region and σ, the magnitude of bulging. It is convenient to introduce a single parameter

$$\varepsilon = \pi\sigma\,(\lambda^4 + \mu^4 + 4\lambda^2\mu^2),$$

which characterizes the postcritical shell deformation. Using the above parameter, the energy of shell deformation and the work done by the external pressure are evaluated from the formulas

$$U = \frac{4E\delta^2 \sqrt{ab}\,\varepsilon}{\sqrt{12}\,(1-\nu^2)},$$
$$A = \frac{p\varepsilon}{\sqrt{ab}}.$$

The load supported by the shell is evaluated by using the shell equilibrium condition at the moment of bulging

$$\frac{d}{d\varepsilon}(U - A) = 0.$$

We have

$$\frac{4E\delta^2 \sqrt{ab}}{\sqrt{12}\,(1-\nu^2)} - \frac{p}{\sqrt{ab}} = 0,$$

and therefore

$$p = \frac{2E\delta^2 ab}{\sqrt{12}\,(1-\nu^2)}.$$

Taking into account the fact that a and b are the principal curvatures of the shell at the center of bulging, we can rewrite the above formula as

$$p = \frac{2E\delta^2}{\sqrt{12}\,(1-\nu^2)\,R_1 R_2},$$

where R_1 and R_2 denote the principal radii of curvature.

We observe that under the more general assumptions regarding the mode of bulging of the shell at the moment of loss of stability, we still [illegible] pressure as in the more simple case studied in section 1.

2. Special isometric transformation of a convex surface of rotation

Experience shows that loss of stability of a shell of rotation of positive Gaussian curvature subjected to internal pressure may occur simultaneously with the formation of regularly placed elliptical dents along some parallel (Figure 6). The physical reason for such a loss of stability is as follows. It is possible that under the above-mentioned mode of shell deformation, accompanied by the formation of dents elongated along the meridians, there may occur a general increase of volume confined within the shell, in spite of the caving in of the shell surface along the system of dents inward into the confined volume.

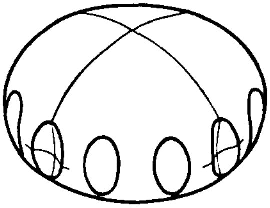

FIGURE 6.

As in our previous study we shall approximate the shape of the shell under bulging to an isometric transformation of the original surface. We shall not study such a transformation in detail, but limit ourselves to evaluating all such quantities related to deformation as will be required for the solution of the shell stability problem. In particular, we are interested in finding out by how much the planes of the parallels enclosing the system of dents move apart.

Since deformation of the shell without the zone of dents is small, the finite surface bending of this part of the shell may be considered as an infinitesimally small bending. The corresponding bending field will be evaluated by superposition of bending fields related to the formation of various single dents. A bending field, conditioned by the formation of a single dent (of the region of bulging) will be considered by us in the form determined in section 2.

Let A be any point of the surface situated at a small distance from the parallel h along which the regions of bulging are situated. Let us evaluate the $\tilde{\xi}$ component of the bending field in the direction of the meridian. Let P be the foot of the geodetic perpendicular drawn from point A to the parallel γ. We introduce a system of rectilinear coordinates x, y, z, in such a way that the x-axis be tangent to the meridian, the y-axis tangent to the parallel, and the z-axis normal to the surface.

It is natural to assume that the magnitude of the $\tilde{\xi}$ component is dependent mainly on the shell bulging regions in the vicinity of point P. Therefore, if we denote by $\xi(x, y)$ the bending field component along the meridian, corresponding to the bulging region having P as a center, then the component $\tilde{\xi}$ in which we are interested, and which is a function of the whole system of bulging regions produced, will be

$$\tilde{\xi} = \sum_k \xi(h, y_k),$$

where y_k denotes the coordinates of the centers of the adjacent bulging regions.

Let us examine more closely the function $\xi(x, y)$. We recall that in addition to the variables x, y, we have also introduced the variables u, v, defined by the relations

$$x\sqrt{a} = u, \quad y\sqrt{b} = v,$$

the complex variable

$$w = u + iv$$

and the complex variable ω

$$w = \lambda\omega + \frac{\mu}{\omega}$$

(see section 2). In the above expressions, a and b are the principal curvatures at the center of bulging P, and λ, μ are parameters characterizing the shape of a bulging region. It is imperative to note that λ and μ are small for small bulging regions.

In section 2 we have derived the following expression for the bending field component ξ, related to the appearance of one single bulging region

$$\xi = \sqrt{a}\,\mathrm{Re}\left(-u\zeta + \int \zeta\, dw\right),$$

where $\zeta(w)$ is an analytic function of the complex variable w which, outside of the region of bulging, is determined with the help of the formula

$$\zeta = \frac{\alpha}{\omega^2}, \quad \alpha = -\frac{3\mu}{\lambda}(\lambda^2 + \mu^2)\sigma.$$

Introducing the same value of ζ in the formula for ξ and integrating, we obtain

$$\xi = \alpha \sqrt{a}\, \mathrm{Re}\left(-\frac{u}{\omega^2} - \frac{\lambda}{\omega} + \frac{\mu}{3\omega^3}\right).$$

When $|w|$ has a lower limit, then the corresponding absolute value of ω, defined by the relation

$$w = \lambda\omega + \frac{\mu}{\omega},$$

is quite big for small λ and μ. It follows, therefore, that for small λ and μ we may neglect the term $\mu/3\omega^3$ in the formula for ξ and, moreover, we can consider that

$$w = \lambda\omega.$$

We may therefore rewrite the formula for the ξ component in a simplified form

$$\xi = -\lambda^2\alpha \sqrt{a}\, \mathrm{Re}\left(\frac{u}{w^2} + \frac{1}{w}\right),$$

or, by separating the real part

$$\xi = -\lambda\alpha \sqrt{\ } a u \frac{}{(u^2 + v^2)^2}.$$

Let

$$v_k = \sqrt{b}\, y_k.$$

Then

$$\tilde{\xi} = -\lambda^2\alpha \sqrt{a}\, u^3 \sum_k \frac{1}{(u^2 + v_k^2)^2},$$

where

$$u = h\sqrt{a}.$$

The summation on the right-hand side of this equality, in accordance with our assumption, includes the adjacent regions. However, because of the very satisfactory convergence of the series, we may consider the summing up as carried out for all values of k:

$$\tilde{\xi} = -\lambda^2\alpha \sqrt{a}\, u^3 \sum_{-\infty}^{\infty} \frac{1}{(u^2 + v_k^2)^2}.$$

When the spacings of the bulging regions are close enough, i.e., when

$$\Delta v = v_k - v_{k-1}$$

is small, we may replace summation by integration in the formula for $\tilde{\xi}$. We then obtain

$$\tilde{\xi} = -\frac{\lambda^2\alpha\sqrt{a}}{\Delta v}u^3\int_{-\infty}^{\infty}\frac{dv}{(u^2+v^2)^2},$$

or, by replacing the variable $v = ut$,

$$\tilde{\xi} = -\frac{\lambda^2\alpha\sqrt{a}}{\Delta v}\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^2}.$$

We have

$$\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^2} = \frac{\pi}{2},$$

and therefore

$$\tilde{\xi} = -\frac{\pi\lambda^2\alpha\sqrt{a}}{2\Delta v}.$$

Introducing in the above the value

$$\alpha = -\frac{3\mu}{\lambda}(\lambda^2+\mu^2)\sigma,$$

we obtain the following final formula for $\tilde{\xi}$:

$$\tilde{\xi} = \frac{3\pi\sqrt{a}\lambda\mu(\lambda^2+\mu^2)\sigma}{2\Delta v}.$$

If we have point A on the opposite side of the zone of bulging regions, it will be displaced along the meridian, under the deformation in question, by exactly the same amount but in the opposite direction. It follows, therefore, that the moving apart of the parallel planes, confining the zone of the bulging regions in which we are interested, will amount to

$$\varepsilon = \frac{3\pi\sqrt{a}\lambda\mu(\lambda^2+\mu^2)\sigma\cos\alpha}{\Delta v},$$

where α is the angle between the tangent to the meridian and the surface axis. Introducing

$$y = \frac{v}{\sqrt{b}},$$

we obtain

$$\varepsilon = \frac{3\pi}{\Delta y}\sqrt{\frac{a}{b}}\lambda\mu(\lambda^2+\mu^2)\sigma.$$

We recall that in the above a and b denote the normal curvatures of the surface in the direction of the meridian and parallel respectively, Δy is the distance between the centers of adjacent bulging regions, and $\lambda\mu(\lambda^2+\mu^2)\sigma$ is a quantity characterizing a separately taken bulging region.

With the evaluation of the quantity ε, we complete our study of the isometrically transformed surface and will proceed to investigate the shell stability problem.

3. Critical internal pressure for a convex shell of rotation

The critical internal pressure causing loss of stability of a shell of rotation, accompanied by the formation of a system of bulging regions along some parallel circle, will be determined by a study of the elastic equilibrium under substantial bulging. The equilibrium condition is

$$d(U-A)=0,$$

where U is the shell deformation energy and A is the work done by the pressure.

In paragraph 1 we arrived at the following expression for the shell deformation energy, related to the formation of one bulging region

$$U=\frac{2Eh^2 2\pi\sqrt{ab}}{\sqrt{12(1-\nu^2)}}\sigma(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

If we denote by n the number of bulging regions, the corresponding expression for the total shell deformation energy will be

$$U=\frac{[illegible]}{\sqrt{12(1-\nu^2)}}\sigma(\lambda^4+\mu^4+4\lambda^2\mu^2)\,n.$$

We now turn to the question of work A. If we denote by ΔV the change in volume enclosed by the shell caused by bulging, then the work done will be

$$A=p\Delta V,$$

where p denotes the internal pressure.

Let us consider two planes perpendicular to the surface axis and confining the zone of bulging regions. The change in volume, ΔV, is conditioned by an increase in distance between the two planes under deformation of the shell and formation of bulging regions. We shall denote the corresponding components of ΔV by ΔV_e and ΔV_i respectively.

The quantity ΔV_i is negative and its value, corresponding to one bulging region, is given by the formula

$$\Delta V_i'=-\frac{\pi\sigma}{\sqrt{ab}}(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

Consequently, for all the n regions

$$\Delta V_i=-\frac{\pi\sigma}{\sqrt{ab}}(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

The other quantity is

$$\Delta V_e = \pi\rho^2\varepsilon,$$

where ρ is the radius of the parallel circle along which the bulging regions are situated, and ε is the increase in distance between the two parallel planes enclosing the zone of bulging regions under the deformation of the shell. Substituting in the above the value of ε as determined in paragraph 2, we obtain

$$\Delta V_e = \pi\rho^2 \frac{3\pi}{\Delta y}\sqrt{\frac{a}{b}}\,\lambda\mu\,(\lambda^2+\mu^2)\,\sigma\cos\alpha.$$

Finally, we obtain the following expression for the work A done by the internal pressure p:

$$A = -\frac{\pi\sigma(\lambda^4+\mu^4+4\lambda^2\mu^2)\,pn}{\sqrt{ab}} + \\ + \pi\rho^2\frac{3\pi}{\Delta y}\sqrt{\frac{a}{b}}\cos\alpha\lambda\mu\,(\lambda^2+\mu^2)\,\sigma p.$$

Fixing the shape of a bulging region (parameters μ and λ) let us vary the deflection in the bulging regions (parameter σ). From the condition of equilibrium

$$\frac{d}{d\sigma}(U-A)=0$$

we obtain the following relation for the value of pressure p supported by the shell at bulging

$$\frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}(\lambda^4+\mu^4+4\lambda^2\mu^2)\,n + \frac{\pi}{\sqrt{ab}}(\lambda^4+\mu^4+4\lambda^2\mu^2)\,np - \\ - \pi\rho^2\frac{3\pi}{\Delta y}\sqrt{\frac{a}{b}}\cos\alpha\lambda\mu\,(\lambda^2+\mu^2)\,p = 0.$$

Multiplying the above relation by

$$\frac{\Delta y}{2\pi^2\rho}\sqrt{ab}$$

and noting that

$$n\Delta y = 2\pi\rho,$$

we obtain

$$\frac{4E\delta^2 ab}{\sqrt{12}\,(1-\nu^2)}(\lambda^4+\mu^4+4\lambda^2\mu^2) + \\ + (\lambda^4+\mu^4+4\lambda^2\mu^2)\,p - \frac{3\rho}{2}\,a\lambda\mu\,(\lambda^2+\mu^2)\,p\cos\alpha = 0.$$

Dividing the above by

$$\lambda^4+\mu^4+4\lambda^2\mu^2$$

and assuming

$$\vartheta = \frac{\lambda\mu}{\lambda^2+\mu^2},$$

leads us to the expression

$$\frac{4E\delta^2 ab}{\sqrt{12(1-\nu^2)}} + p - \frac{3a\rho\cos\alpha}{2}\,\frac{\vartheta p}{1+2\vartheta^2} = 0.$$

Therefore,

$$p = \frac{4E\delta^2 ab}{\sqrt{12(1-\nu^2)}}\,\frac{1}{\frac{3a\rho\cos\alpha}{2}\,\vartheta^* - 1},$$

where

$$\vartheta^* = \frac{\vartheta}{1+2\vartheta^2}.$$

The parameter ϑ^* is a function of the parameters λ, μ and, consequently, is a characteristic of the shape of a bulging region. Let us investigate the domain of allowable values of the parameter ϑ^*. To do this we note first of all that

$$\vartheta = \frac{\lambda\mu}{\lambda^2+\mu^2}$$

has $-1/2$ and $+1/2$ as its limits. Further, ϑ^* is a monotonic function of ϑ, since

$$\frac{d\vartheta^*}{d\vartheta} = \frac{1-2\vartheta^2}{(1+2\vartheta^2)^2} > 0.$$

It follows, therefore, that ϑ^* has $-1/3$ and $+1/3$ as its limits.

Taking into account the interval of allowable values of ϑ^* we conclude that the least pressure under which the shell may lose its stability, this being accompanied by bulging along a given parallel circle, is given by the formula

$$p = \frac{2E\delta^2 ab}{\sqrt{3(1-\nu^2)}}\,\frac{1}{\frac{a\rho}{2}\cos\alpha - 1}.$$

It should be remembered that in the above formula, a and b are the normal curvatures of the surface along the meridian and parallel respectively, ρ is the radius of the parallel, and α is the angle between the tangent to the meridian and the surface axis. Introducing in this formula R_1 and R_2, the principal radii of curvature of the surface, where

$$R_1 = \frac{1}{a}, \quad R_2 = \frac{1}{b}$$

and remembering that

$$\frac{\cos\alpha}{\rho} = \frac{1}{R_2},$$

we obtain

$$p = \frac{2E\delta^2}{\sqrt{3(1-\nu^2)}\,R_1R_2}\,\frac{1}{\frac{\rho^2}{2R_1R_2} - 1}.$$

The least value of p is obtained by letting $\vartheta^* = {}^1/_3$. The corresponding value of ϑ is ${}^1/_2$. Since

$$\vartheta = \frac{\lambda\mu}{\lambda^2+\mu^2},$$

this is only possible when $\lambda = \mu$, meaning that the bulging region defined by the equations

$$x = \frac{\lambda+\mu}{\sqrt{a}}\cos\varphi, \qquad y = \frac{\lambda-\mu}{\sqrt{b}}\sin\varphi,$$

degenerates into a segment of the x-axis (meridian). The physical interpretation of such a result is that the dents formed as a result of loss of stability by the shell must be greatly elongated along the meridians. This has been confirmed by corresponding experiments.

Let us use the above formula to evaluate the critical pressure for a flattened out ellipsoid of rotation. Let a and b be the semiaxes of the ellipsoid, where $b < a$. Since the Gaussian curvature of a flattened out ellipsoid increases monotonically as the equator is approached, and the radius, ρ, of the parallel increases as well, the minimal value of p is obtained when bulging occurs along the equator. On the equator

$$\rho = a, \quad R_2 = a, \quad R_1 = \frac{b^2}{a}.$$

It follows that

$$p = \frac{2E\delta^2}{\sqrt{3}\,(1-\nu^2)}\,\frac{1}{\frac{a^2}{2}-b^2}.$$

In the case of a strongly flattened out ellipsoid $(b \ll a)$

$$p \approx \frac{4E\delta^2}{\sqrt{3}\,(1-\nu^2)\,a^2}.$$

It is important to note that the magnitude of the critical pressure is never lower than the above value, for any degree of surface flattening out.

4. Loss of stability of a convex shell of rotation subjected to external pressure

In section 1, and in paragraph 1 of this section, we discussed the problem of loss of stability of a shallow convex shell subjected to external pressure. The following formula for the value of critical pressure was obtained

$$p = \frac{2E\delta^2}{\sqrt{3}(1-\nu^2)\,R_1R_2}.$$

The above result was based on the assumption that the shell is shallow and in calculating the work done by external pressure

$$A = p\Delta V,$$

we used the following expression for ΔV, the change in volume confined by the shell,

$$\Delta V = \iint \zeta dxdy,$$

where ζ is the z component of the bending field during shell deformation. This formula would be exact if the bending field were to be perpendicular to the xy plane, and accordingly, the deformed surface were to be defined by

$$z = z_0 + \zeta,$$

where z_0 is related to the original surface. In reality, such an assumption may be considered to hold good only in the vicinity of the center of bulging and, consequently, we might expect a different value of the external critical pressure for a nonshallow shell. Presently we shall discuss this problem for convex shells of rotation.

We assume that loss of stability of a convex shell of rotation loaded by an external pressure is accompanied by the formation of a system of dents along some parallel (Figure 7). Comparison with the result obtained in paragraph 2 leads us to think that such a loss of stability can be realized if the dents are substantially elongated in the direction of the parallel along which they are situated. As in the case of internal pressure, we shall consider the change in volume confined by the shell under deformation as being made up of two parts: ΔV_i and ΔV_e. ΔV_i is the decrease in volume directly related to the formation of bulging regions, and ΔV_e is determined by the proximity of the parallel planes containing the zone of bulging regions:

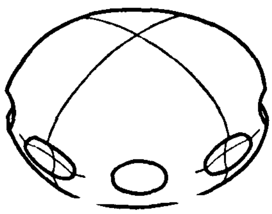

FIGURE 7.

$$\Delta V_i = \frac{\pi\sigma}{\sqrt{ab}} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2) n,$$

$$\Delta V_e = -\pi\rho^2 \frac{3\pi}{\Delta y} \sqrt{\frac{a}{b}} \lambda\mu (\lambda^2 + \mu^2) \sigma \cos\alpha.$$

It follows that work done by external pressure, p, under shell deformation is equal to

$$A = \frac{\pi\sigma}{\sqrt{ab}} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2) pn +$$
$$+ \pi\rho^2 \frac{3\pi}{\Delta y} \sqrt{\frac{a}{b}} \cos\alpha\lambda\mu (\lambda^2 + \mu^2) \sigma p.$$

This formula differs only in sign from the corresponding formula in the case of internal pressure.

As far as energy of deformation is concerned, its value is given by the expression derived previously, namely

$$U = \frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}(1-\nu^2)}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\,\sigma n.$$

As in the case of internal pressure, from the condition of shell equilibrium,

$$\frac{d}{d\sigma}(U - A) = 0,$$

we obtain the following relation for the value of pressure, p, supported by the shell at bulging:

$$\frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}(1-\nu^2)}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\,n -$$
$$-\frac{\pi}{\sqrt{ab}}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\,np +$$
$$+\pi\rho^2\frac{3\pi}{\Delta y}\sqrt{\frac{a}{b}}\cos\alpha\lambda\mu(\lambda^2 + \mu^2)\,p = 0.$$

Solving for p, we obtain the following value after simplification:

$$p = \frac{4E\delta^2 ab}{\sqrt{12}(1-\nu^2)}\,\frac{1}{-\frac{3ap\cos\alpha}{2}\vartheta^* + 1},$$

where, as before,

$$\vartheta^* = \frac{\vartheta}{1 + 2\vartheta^2},$$

and

$$\vartheta = \frac{\lambda\mu}{\lambda^2 + \mu^2}.$$

The least value of p is obtained for the greatest, in absolute magnitude, negative value of ϑ^*, i.e., for $\vartheta^* = -1/_3$. Substituting this in the expression for p, we obtain

$$p = \frac{4E\delta^2 ab}{\sqrt{12}(1-\nu^2)}\cdot\frac{1}{\frac{ap}{2}\cos\alpha + 1}.$$

Application of this formula to the case of shallow shells, if this is at all possible, results in a value of p which differs but slightly from that obtained previously

$$p = \frac{2E\delta^2 ab}{\sqrt{3}(1-\nu^2)},$$

since in the case of shallow shells, $\alpha \simeq \pi/2$, and, consequently, $\cos\alpha \simeq 0$.

Let us examine the shape of the bulging regions. Since $\vartheta^* = -1/_3$, it follows that $\vartheta = -1/_2$ and, consequently, $\lambda = -\mu$. A bulging region is defined by the equation

$$x=\frac{\lambda+\mu}{\sqrt{a}}\cos\varphi,\qquad y=\frac{\lambda-\mu}{\sqrt{b}}\sin\varphi.$$

When $\lambda=-\mu$, our ellipse degenerates into a segment of the y-axis. The physical interpretation of this is that at loss of stability caused by external pressure, the regions of bulging are strongly elongated along the parallel.

As in the case of internal pressure, the formula for the critical load can be transformed as follows:

$$p=\frac{4E\delta^2}{\sqrt{12(1-\nu^2)}\,R_1R_2}\,\frac{1}{\frac{\rho^2}{2R_1R_2}+1},$$

or,

$$p=\frac{2E\delta^2}{\sqrt{3(1-\nu^2)}}\cdot\frac{1}{\frac{\rho^2}{2}+R_1R_2}.$$

It should be remembered that in the above, R_1 and R_2 denote the principal normal curvatures of the shell along the parallel where bulging takes place, and ρ is the radius of the parallel.

As an application of the above result we shall consider loss of stability of a closed spherical shell of radius R. In this case,

$$R_1=R,\quad R_2=R.$$

The minimum value of p is obtained for $\rho=R$, i.e., when formation of dents takes place along the equator. The corresponding formula for the critical pressure is

$$p=\frac{2E\delta^2}{\sqrt{3(1-\nu^2)}R^2}\,\frac{2}{3}.$$

This value equals $^2/_3$ of the corresponding value for shallow shells.

Remark. According to data obtained from experiments carried out with spherical segments and described in the book by A. S. Vol'mir,* bulging under external pressure commences at the edge of the segment. It is reasonable to assume that this is accompanied by loss of stability, as described in the present section.

5. Loss of stability of shells of rotation subjected to torsion

A shell of rotation subjected to the action of a turning moment applied at its edge may lose its stability with formation of bulging regions inclined to the meridian (Figure 8). Let us find the value of a torque causing loss of stability in such a case.

* Vol'mir, A. S. Gibkie plastinki i obolochki (Flexible Plates and Shells). — Gostekhizdat. 1956.

Approximating the deformed shell surface to an isometric transformation of its original shape, we shall use the same considerations as in paragraph 2 for the case of internal pressure. In this case there occurs a certain twist of the shell, through an angle ε, as evidenced by the bulging regions inclined to the meridian. Let us evaluate the magnitude of this angle.

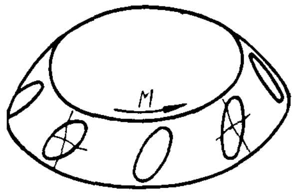

FIGURE 8.

We take any point A on the shell surface situated outside the zone of bulging and in its vicinity. Let us find the displacement of this point along the parallel, caused by the deformation in question. Consider a perpendicular drawn from point A to the parallel γ, along which the centers of bulging regions are situated, and let P denote the foot of this perpendicular. As in paragraph 2, we introduce a system of rectilinear coordinates taking point P as the origin, the tangent plane at P to be the xy plane, and directing the x-axis along the meridian of the surface.

Let $\tilde{\eta}$ be the displacement of point A in which we are interested. Assuming that the magnitude of $\tilde{\eta}$ is mainly dependent only on the regions of bulging situated in the vicinity of P, we can state:

$$\tilde{\eta} = \sum_k \eta(h, y_k).$$

In the above, $\eta(x, y)$ is the y component of the bending field corresponding to the bulging region having P as its center, h is the distance between point A and the parallel γ, and y_k are the coordinates of the centers of bulging regions which are near to P.

Let us examine the function $\eta(x, y)$. It can be expressed with the help of the analytic function

$$\zeta = \frac{\alpha}{\omega^2}$$

in accordance with the formula

$$\eta = \alpha\sqrt{b}\,\mathrm{Re}\left\{-(u\sin\vartheta + v\cos\vartheta) - ie^{i\vartheta}\int\zeta dn\right\}.$$

Substituting $\zeta = \alpha/\omega^2$ in the above formula, and noting that,

$$w = \lambda\omega + \frac{\mu}{\omega},$$

we obtain

$$\eta = \alpha\sqrt{b}\,\mathrm{Re}\left\{-\frac{(u\sin\vartheta + v\cos\vartheta)}{\omega^2} + \frac{i\lambda e^{i\vartheta}}{\omega} - \frac{i\mu e^{i\vartheta}}{\omega^3}\right\}.$$

For small bulging regions $|\omega|$ is large. Consequently we may consider that

$$\eta = \alpha\sqrt{b}\,\mathrm{Re}\left\{-\frac{(u\sin\vartheta + v\cos\vartheta)}{\omega^3} + \frac{i\lambda e^{i\vartheta}}{\omega}\right\}.$$

For the same section

$$w \simeq \lambda\omega,$$

and therefore,

$$\eta = \alpha\lambda^2\sqrt{b}\,\mathrm{Re}\left\{-\frac{u\sin\vartheta + v\cos\vartheta}{w^3} + \frac{ie^{i\vartheta}}{w}\right\}.$$

Let us make a transition from the coordinates u, v to x, y. We have,

$$x = \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),$$
$$y = \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta),$$
$$x\sqrt{a} + iy\sqrt{b} = (u + iv)\,e^{i\vartheta} = we^{i\vartheta},$$
$$ax^2 + by^2 = u^2 + v^2 = |w|^2.$$

With the help of the above relations, the expression for η can be transformed as under

$$\eta = 2\lambda^2\alpha\sqrt{b}\,\frac{2y^3b\sqrt{b}\cos 2\vartheta - (3xy^2b\sqrt{a} + x^3a\sqrt{a})\sin 2\vartheta}{(ax^2 + by^2)^3}.$$

Substituting this value in the expression for $\tilde{\eta}$, we obtain

$$\tilde{\eta} = 2\lambda^2\alpha\sqrt{b}\sum_k \frac{2y_k^3b\sqrt{b}\cos 2\vartheta - (3hy_k^2b\sqrt{a} + h^3a\sqrt{a})\sin 2\vartheta}{(ah^2 + by_k^2)^3}.$$

Because of the symmetry of the bulging regions with respect to the point A, we may omit summation with respect to the first term and take

$$\tilde{\eta} = -2\lambda^2\alpha\sqrt{b}\sin 2\vartheta\sum_k \frac{3hy_k^2b\sqrt{a} + h^3a\sqrt{a}}{(ah^2 + by_k^2)^3}.$$

Further, as in paragraph 2, we replace summation on the right-hand side by integration. Denoting

$$\Delta y = y_k - y_{k-1},$$

we shall have

$$\tilde{\eta} = -\frac{2\lambda^2\alpha\sqrt{b}\sin 2\vartheta}{\Delta y}\int_{-\infty}^{\infty}\frac{3hy^2b\sqrt{a} + h^3a\sqrt{a}}{(ah^2 + by^2)^3}\,dy.$$

Introducing a new variable t, we obtain

$$y = h\sqrt{\frac{a}{b}}\,t,$$

$$\tilde{\eta} = -\frac{2\lambda^2\alpha\sin 2\vartheta}{\Delta y}\int_{-\infty}^{\infty}\frac{3t^2 + 1}{(1 + t^2)^3}\,dt.$$

$$\int_{-\infty}^{\infty}\frac{3t^2 + 1}{(1 + t^2)^3}\,dt = 2\pi.$$

Therefore

$$\tilde{\eta} = -\frac{4\pi\lambda^2\alpha \sin 2\vartheta}{\Delta y}.$$

Introducing in the above

$$\alpha = -\frac{3\mu}{\lambda}(\lambda^2 + \mu^2)\sigma,$$

we obtain a final formula for $\tilde{\eta}$

$$\tilde{\eta} = \frac{12\pi \sin 2\vartheta\lambda\mu(\lambda^2 + \mu^2)\sigma}{\Delta y}.$$

As was to be expected, the displacement, $\tilde{\eta}$, of point A is independent of the distance h.

We are now in a position to determine the value of the angle of twist ε. If we denote by ρ the radius of the parallel γ along which the regions of bulging are situated, then

$$\varepsilon = \frac{2\tilde{\eta}}{\rho} = \frac{24\pi \sin 2\vartheta\lambda\mu(\lambda^2 + \mu^2)\sigma}{\rho\,\Delta y}.$$

The work done by moment M, at bulging, is

$$A = M\varepsilon = \frac{24\pi \sin 2\vartheta\lambda\mu(\lambda^2 + \mu^2)\sigma M}{\rho\,\Delta y}.$$

Let us find the shell deformation energy, U. In section 1 we derived the following formula for the energy of shell deformation, calculated per unit length of the boundary of bulging:

$$\bar{U} = \frac{2E\delta^2\alpha hk}{\sqrt{12(1-\nu^2)}}.$$

In the above formula, α is the angle between the plane of contact of curve γ, which is the boundary of bulging zone, and the planes to the surface; k is the curvature of curve γ; and h is the deflection within the zone of bulging.

Referred to the system of coordinates introduced previously, the surface in the vicinity of point P is defined by the equation

$$z = \frac{1}{2}(ax^2 + by^2).$$

The curve γ on the surface is given by the equations

$$u = (\lambda + \mu)\cos\varphi, \quad v = (\lambda - \mu)\sin\varphi.$$

Variables u, v are related to x, y by the formulas

$$x = \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),$$

$$y = \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta).$$

For small bulging regions we may consider that

$$\alpha = \frac{k_n}{k},$$

where k is the curvature of curve γ, and k_n is the normal curvature of the original surface in the γ direction. A general expression for the normal curvature is

$$k_n = \frac{a dx^2 + b dy^2}{dx^2 + dy^2}.$$

Substituting in the above the values

$$x = \frac{1}{\sqrt{a}} \{(\lambda + \mu) \cos\varphi \cos\vartheta - (\lambda - \mu) \sin\varphi \sin\vartheta\},$$
$$y = \frac{1}{\sqrt{b}} \{(\lambda + \mu) \cos\varphi \sin\vartheta + (\lambda - \mu) \sin\varphi \cos\vartheta\},$$

we obtain the normal curvature of the surface in the γ direction

$$k_n = \frac{a x'^2 + b y'^2}{x'^2 + y'^2},$$

differentiation being carried out with respect to the variable φ.

In the case of a small bulging region, the curvature of γ may be evaluated with the help of its projection on the xy plane. We obtain

$$k = \frac{|x'' y' - y'' x'|}{(x'^2 + y'^2)^{3/2}}.$$

An element of arc of curve γ is given by

$$ds = (x'^2 + y'^2)^{1/2} d\varphi.$$

Let us evaluate the integral

$$\int_\gamma \alpha^2 k \, ds = \int_0^{2\pi} \frac{(a x'^2 + b y'^2)^2}{|x'' y' - y'' x'|} d\varphi.$$

We have

$$|x'' y' - y'' x'| = \frac{1}{\sqrt{ab}} (\lambda^2 - \mu^2),$$
$$a x'^2 + b y'^2 = (\lambda^2 + \mu^2) \sin^2\varphi + (\lambda - \mu)^2 \cos^2\varphi.$$

Substituting the above values in the expression under the integral sign, we obtain

$$\int_\gamma \alpha^2 k \, ds = \frac{\sqrt{ab}}{\lambda^2 - \mu^2} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

The full energy of deformation is

$$U = \int\limits_{\gamma} \overline{U}\, ds_{\gamma} = \frac{2E\delta^2 h}{\sqrt{12}\,(1-\nu^2)}\,\frac{2\pi\sqrt{ab}}{\lambda^2-\mu^2}\,(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

Substituting in the above

$$h = (\lambda^2 - \mu^2)\,\sigma,$$

we obtain

$$U = \frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(\lambda^4+\mu^4+4\lambda^2\mu^2)\,\sigma.$$

The energy of deformation for all the n regions of bulging is

$$U = \frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(\lambda^4+\mu^4+4\lambda^2\mu^2)\,\sigma n.$$

From the condition of equilibrium of the shell

$$\frac{d}{d\sigma}(U - A) = 0$$

we obtain an expression for the moment M which causes loss of stability of the shell

$$\frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(\lambda^4+\mu^4+4\lambda^2\mu^2)\,n - \frac{24\pi\sin 2\vartheta\lambda\mu\,(\lambda^2+\mu^2)\,M}{\rho\,\Delta y} = 0.$$

Noting that

$$n\,\Delta y = 2\pi\rho,$$

we obtain

$$\frac{\pi\rho^2 E\delta^2\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(1+2\varepsilon^2) - 3\sin 2\vartheta\,\varepsilon M = 0,$$

where

$$\varepsilon = \frac{\lambda\mu}{\lambda^2+\mu^2}.$$

The least value of M is obtained for $\varepsilon = \frac{1}{2}$ and $\vartheta = 45$. This value is evaluated with the help of the formula

$$M = \frac{\pi\rho^2 E\delta^2\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)},$$

or,

$$M = \frac{\pi\rho^2 E\delta^2}{\sqrt{12}\,(1-\nu^2)\sqrt{R_1 R_2}}.$$

In the above, R_1 and R_2 are the principal radii of curvature along the parallel where bulging takes place, and ρ is the radius of the parallel.

In conclusion let us note that loss of stability under the action of a twisting moment M is accompanied by formation of strongly elongated dents ($\varepsilon = 1/_2$) inclined to the meridian at an angle of $\vartheta = 45°$.

§ 4. POSTCRITICAL DEFORMATIONS OF SHELLS OF POSITIVE GAUSSIAN CURVATURE SUBJECTED TO EXTERNAL PRESSURE. INFLUENCE OF INITIAL DEFLECTION ON STABILITY OF SHELLS

The subject of postcritical deformations of shells of positive Gaussian curvature subjected to external pressure or to the action of a concentrated load was studied by the author in /2/. The present study differs in that it offers a more exact expression for the value of the elastic deformation energy. Results obtained are used for investigating the influence of initial shell deflection on its stability. The value of the working load is determined for shells subjected to external pressure.

1. The simplest postcritical deformation

In the study carried out in /2/ of postcritical deformations of shells of positive Gaussian curvature rigidly fixed at the edges, we approximated the deformed shell surface by a mirror image bulging. Such an assumption was based on the fact that internal deformations of the shell middle surface are small and [illegible] must be considered, to some approximation, as a geometric bending. By identifying the shell postcritical deformation with an isometric transformation we have arrived, through purely geometric considerations, at the conclusion that deformation of the shell surface must be close enough to a corresponding shape of a mirror image bulging.

The decisive criterion in the above derivation was the assumption that deformation of the shell middle surface constitutes a geometric bending, and the further assumption that the shell is rigidly fixed at the edges. In reality, the above conditions are only partly satisfied, especially when we speak of a rigidly fixed surface edge. Although rigidity of a fixed edge may not be absolute, in the case of an elastic shell the significance of a rigidly fixed edge decreases as we move further away from the edge and, hence, its force as an argument of proof is lost in the reality of a practical case.

The above consideration limits application of the result arrived at in /2/, regarding the approximation of postcritical deformation by a mirror image bulging, to the case when the region of bulging encompasses the greater part of the shell and the rigidly fixed shell edge is near to the boundary of this region.

In the case of a shell where the above conditions are not satisfied, namely, the rigidity of the fixed edge is not great enough or, the size of the bulging region is small compared to the dimensions of the shell, thus violating the stipulation regarding the nearness of the built-in edge, we shall approximate the shape of the shell by a general isometric transformation.

We shall name the case of postcritical deformation of a shell of positive Gaussian curvature, as the simplest, if it can be satisfactorily approximated by a mirror image bulging. Where such an approximation is not possible, we shall name the case as general. Study /2/ discusses the simplest cases of postcritical deformations of a shell of positive Gaussian curvature when subjected to a uniform external pressure or to the action of a concentrated load. Presently we wish to deal with this question again, in connection with the more exact expression for shell deformation energy in the zone of strong local bending on the boundary of bulging /1/.

As always, we split the energy of elastic deformation of the shell into two parts, U_γ and U_F. U_γ is the energy of elastic deformation in the zone of strong local bending along the ribs, and U_F is the energy of bending across the main surface of the shell.

The value of energy U_F, per unit area of shell surface, is determined using the formula

$$\overline{U}_F = \frac{E\delta^3}{6(1-\mu^2)}(k_1^2 + k_2^2 + 2\mu k_1 k_2),$$

where k_1 and k_2 are the principal shell curvatures; δ is the shell thickness; E the modulus of elasticity; and μ, Poisson's ratio.

The energy of deformation in the zone of strong local bending along the ribs of the surface, which come into being in the process of postcritical deformation, was evaluated in /2/, and per unit length of a rib is given by the formula

$$\overline{U}_\gamma = cE\delta^{5/2}\alpha^{5/2}k^{1/2}.$$

In the above formula, α is the angle between the plane of contact of the rib and the tangent planes of the shell surface; k is the rib curvature; δ, the shell thickness; and the constant $c \simeq 0.2$. A more exact expression for the energy of deformation in the zone of strong local bending was derived in /1/. It differs from the expression given above by the quantity

$$\Delta\overline{U}_\gamma = \frac{E\delta^3\alpha}{6(1-\mu^2)}\left(-\frac{1}{R} + \frac{k_i + k_e}{2}\right),$$

where $1/R$ is the normal curvature of the original surface in a direction perpendicular to the rib, and k_i and k_e are the normal curvatures of the deformed surface in the same direction, for the inner and outer semi-regions of the rib respectively. Thus, the new expression for the energy of deformation in the zone of strong local bending, and per unit length of a rib, is as follows:

$$\overline{U}_\gamma = cE\delta^{5/2}\alpha^{5/2}k^{1/2} + \frac{E\delta^3\alpha}{6(1-\mu^2)}\left(-\frac{1}{R} + \frac{k_i + k_e}{2}\right).$$

Let us evaluate the energy of postcritical deformation in the simplest case, assuming that the zone of bulging is known to be small. In /2/, the following expressions were derived for the value of the energy:

$$U_\gamma = \pi c E (2h)^{3/2} \delta^{5/2} (k_1 + k_2),$$
$$U_F = \frac{\pi h E \delta^3}{3(1-\mu^2)\sqrt{k_1 k_2}} (k_1^2 + k_2^2 + 2\mu k_1 k_2),$$

where $2h$ is the deflection of the shell at the center of bulging, all other quantities retaining their previous meanings. Taking into account the correction in the value of $\bar{U}_\gamma$, the corresponding difference in the value of U_γ will be

$$\Delta U_\gamma = \frac{E\delta^3}{6(1-\mu^2)} \int_\gamma \alpha \left(-\frac{1}{R} + \frac{k_i + k_e}{2}\right) ds.$$

Let us evaluate this quantity.

First of all we note that k_e and k_i are equal in their absolute values but differ in sign. Therefore,

$$\Delta U_\gamma = -\frac{E\delta^3}{6(1-\mu^2)} \int_\gamma \frac{\alpha\, ds}{R}.$$

The original surface is defined by the equation

$$z = \frac{1}{2}(k_1 x^2 + k_2 y^2),$$

and the zone of bulging is

$$\frac{1}{2}(k_1 x^2 + k_2 y^2) \leqslant h.$$

It is bounded by an ellipse with semiaxes

$$a = \sqrt{\frac{2h}{k_1}}, \quad b = \sqrt{\frac{2h}{k_2}}.$$

It is convenient to define this ellipse in parametric form,

$$x = a \cos t, \quad y = b \sin t.$$

We have

$$\alpha^2 \simeq z_x^2 + z_y^2 = (k_1 x)^2 + (k_2 y)^2.$$

The angular coefficients of the normal to the boundary of bulging

$$\frac{1}{2}(k_1 x^2 + k_2 y^2) = h$$

are equal to $k_1 x$ and $k_2 y$. It follows that the normal curvature of the surface in a direction perpendicular to the boundary of bulging is

$$\frac{1}{R} = k_1 \frac{(k_1 x)^2}{(k_1 x)^2 + (k_2 y)^2} + k_2 \frac{(k_2 y)^2}{(k_1 x)^2 + (k_2 y)^2}.$$

Let us now find an expression for ΔU_{T}, making use of the parametric representation of the ellipse given above. We have,

$$\frac{a}{R}\,ds = \frac{(k_1^3x^2 + k_2^3y^2)}{k_1^2x^2 + k_2^2y^2}\,(k_1^2x^2 + k_2^2y^2)^{1/2}\,ds.$$

Substituting in the above

$$x = \sqrt{\frac{2h}{k_1}}\cos t,\ \ y = \sqrt{\frac{2h}{k_2}}\sin t$$

and

$$ds^2 = 2h\left(\frac{\sin^2 t}{k_1} + \frac{\cos^2 t}{k_2}\right)dt^2,$$

we obtain

$$\frac{a\,ds}{R} = \frac{2h}{\sqrt{k_1k_2}}\,(k_1^2\cos^2 t + k_2^2\sin^2 t)\,dt.$$

Therefore,

$$\Delta U_{\text{T}} = -\frac{\pi hE\delta^3}{3\,(1-\mu^2)}\,\frac{k_1^2 + k_2^2}{\sqrt{k_1k_2}}\,.$$

Comparing this quantity with the energy of bending across the original surface

$$U_F = \frac{\pi hE\delta^3}{3\,(1-\mu^2)\sqrt{k_1k_2}}\,(k_1^2 + k_2^2 + 2\mu k_1k_2);$$

when $\mu = 0$, they differ in sign only.

We assume that a more detailed investigation regarding the value of ΔU_{T}, must bring about a similar conclusion $(U_F = -\Delta U_{\text{T}})$ in the general case as well, i.e., for $\mu \neq 0$. With due regard to this fact, we shall assume in future that $U_F = -\Delta U_{\text{T}}$, and will use the following expression for the full energy of deformation

$$U = \pi cE\,(2h)^{3/2}\delta^{5/2}\,(k_1 + k_2).$$

2. Investigation of postcritical deformations for uniform external pressure and for a concentrated load — the simplest case

In accordance with the general remark made in paragraph 1, we shall discuss now the case when the greater part of the shell is subjected to postcritical deformations, the edge of the shell being rigidly fixed.

Under such conditions we may regard the postcritical deformation of the shell as being close to the corresponding shape of mirror image bulging.

We shall consider first the deformation of the shell under the action of a concentrated load P, acting normally to the shell surface. Denoting the deflection of the shell at the point of application of the force by $2h$, we obtain the following expression for the energy of elastic deformation:

$$U = \pi c E\,(2h)^{3/2}\delta^{5/2}\,(k_1 + k_2).$$

The work done by force P equals

$$A = 2hP.$$

The shell equilibrium condition is characterized by the stationary value of the expression $U - A$, i.e.,

$$d\,(U - A) = 0.$$

From the above expression we deduce the relationship between the force P acting on the shell and the deflection, $2h$, caused by it,

$$P = \frac{3}{2}\pi c E\,\sqrt{2h}\,\delta^{5/2}\,(k_1 + k_2),$$

or,

$$P = 3\pi c E \delta^{5/2} H \sqrt{2h},$$

where H denotes the mean curvature of the shell surface at the point of application of the force.

This formula shows that with the increase of deflection $(2h)$, the force P also increases. This indicates stability of the states of elastic equilibrium when the shell is acted upon by a concentrated load.

It is obvious that application of the above formula is limited by elastic deformations of the shell, i.e., the maximum stresses, conditioned by the deformation in question, must not exceed the elastic limit of the material.

It was shown in /2/ that in the case of the simplest postcritical deformation maximum stress due to bending on the boundary of the bulging zone is determined with the help of the formula

$$\sigma = c'E\,(2h)^{1/2}\delta^{1/2}\sqrt{K},$$

where K is the Gaussian curvature of the surface $(K = k_1 k_2)$, is a constant $\cong 1$, and all other quantities retain their previous meanings. It follows, therefore, that application of the above result is limited to such deflections $2h$, for which

$$c'E\sqrt{2h}\,\delta^{1/2}\sqrt{K} \leqslant \sigma_B,$$

where σ_B is the temporary resistance of the material.

From the above it is possible to determine the force P_e which is certain to produce plastic deformations on the boundary of bulging

$$P_e = \frac{3\pi c}{c'}\,\frac{H}{\sqrt{K}}\,\sigma_B.$$

In particular, for a spherical shell

$$P_e = \frac{3\pi c}{c'}\,\delta^2\sigma_B.$$

It is interesting to note that this force does not depend on the radius of the shell.

Let us investigate now the case when the shell is loaded by a uniform external pressure p. The work done by the external load is

$$A = \frac{2\pi h^2 p}{\sqrt{K}} /2/.$$

From the shell equilibrium condition

$$d(U - A) = 0$$

we obtain the following value for the pressure, p, supported by the shell when bulging rise is $2h$:

$$p = 3cEH\sqrt{K}\frac{\delta^{5/2}}{\sqrt{2h}}.$$

From this formula we see that load p, supported by the shell, decreases as the deflection $(2h)$ increases. This indicates instability of postcritical deformations under uniform loading.

In this case transition to postcritical deformations is effected in an accentuated fashion. Postcritical deformation is being brought to a stop either by the edge of the shell, or by the appearance of plastic deformations on the boundary of bulging. The reasons for stoppage of deformation in the second case, are explained more fully in /1, supplement 1/.

Let us assume that postcritical deformation is brought to a stop by the shell edge against which, ultimately, the region of bulging rests. In such a case the quantity p_i, which is the lowest value of the load supported by the shell, is determined with the help of the maximum allowable deflection, and consequently, by the shell dimensions.

In order to clarify ideas, let us assume that the shell is reinforced by rigid elements running along the lines of curvature of the surface. Such elements divide the shell into rectangular panels with sides a and b. It is reasonable to inquire what the spacing of the reinforcing elements should be in order that the value of the load supported by the shell should not be less than a given quantity p_i.

For the solution of the above problem we determine first the value of the maximum allowable deformation, $2h$, using the condition

$$p_i = 3cEH\sqrt{K}\frac{\delta^{5/2}}{\sqrt{2h}}.$$

We then find the corresponding values of a and b:

$$a = \sqrt{\frac{2h}{k_1}},\quad b = \sqrt{\frac{2h}{k_2}},$$

or

$$a = \frac{3cEH\sqrt{k_2}\,\delta^{5/2}}{p_i},$$
$$b = \frac{3cEH\sqrt{k_1}\,\delta^{5/2}}{p_i}.$$

It is evident that when using the above result, it is not possible to assign too great values to p_i for, in such a case, the corresponding calculated values of a and b would be too small. This would mean that the deformations in question, though they may embrace the entire panel, could not be considered as substantial, in the sense required by the theory.

In order to present the above remark in a more concrete form, let us note that when k_1 and k_2 are of the same order, the theory imposes the observance of the condition

$$\frac{2h}{\delta} \gg 1 \ /1, 2/.$$

In the case under consideration this would mean that

$$\frac{3cEH\sqrt{k}\delta^3}{p_i} \gg 1.$$

Let us find the maximum dimensions of the panel a, b by stipulating that the postcritical deformation stays elastic up to the panel boundary. The consequent maximum deformation $2h$ then satisfies the condition

$$c'E\sqrt{2h}\delta^{1/2}\sqrt{K} = \sigma_B.$$

Substituting the above value of $2h$ into the expressions for a and b, we obtain

$$a = \frac{\sigma_B}{c'E\delta^{1/2}k_1\sqrt{k_2}},$$
$$b = \frac{\sigma_B}{c'E\delta^{1/2}k_2\sqrt{k_1}}.$$

Let us assume that postcritical deformation has been brought to a stop because of the appearance of plastic deformations on the boundary of bulging. We shall evaluate the minimum load supported by the shell in such a case. The deflection, corresponding to such a load, is obtained from the condition

$$c'E\sqrt{2h}\delta^{1/2}\sqrt{K} = \sigma_B.$$

Substituting the above in the formula for p, we obtain

$$p_i = 3cc'\left(\frac{E}{\sigma_B}\right)EHK\delta^3.$$

Application of the above formula is limited by two conditions. First, this method can only be applied if the deformation at the moment when plastic deformation appears on the bulging boundary is substantial. This means that the quantity $2h$, determined with the help of condition

$$c'E\sqrt{2h}\delta^{1/2}\sqrt{K} = \sigma_B$$

must satisfy the inequality

$$\frac{2h}{\delta} \gg 1,$$

or, what amounts to the same thing,

$$\frac{\sigma_B}{c'E\delta\sqrt{K}} \gg 1.$$

Further, with the appearance of plastic deformations on the bulging boundary, the postcritical deformation must embrace a substantial portion of the shell. This means that the linear dimensions a and b of the panel must be of the same order as the quantities

$$a_e = \frac{\sigma_B}{c'E\delta^{1/2}k_1\sqrt{k_2}}, \quad b_e = \frac{\sigma_B}{c'E\delta^{1/2}k_2\sqrt{k_1}}.$$

3. Influence of initial deflection of the shell on its stability. Working load

Experience shows that a convex shell subjected to external pressure loses its stability and commences to bulge at a pressure which is usually less than the critical pressure, and its value is given by the formula

$$p = \frac{2E}{\sqrt{3(1-\nu^2)}} \frac{\delta^2}{R_1 R_2}.$$

The basic reason for such a lowering in the value of the critical pressure is the imperfectness of shape of the actual shell, or, stated otherwise, initial deflection. Experiments carried out with spherical segments prepared to perfect shape substantiate the theoretical value of the critical pressure. There is no reason, therefore, to doubt the validity of the critical pressure formula and no need to attempt to improve it.

The fact that actual critical pressure for a working shell having an initial deflection may possibly be much less than the theoretical, accounts for serious difficulties in shell design, for such a value cannot be accepted as the working pressure. A natural way out of such a situation would be to accept as working load the value of the lower critical load (see paragraph 2). Such a load is conditioned by substantial shell deformations and, therefore, is less sensible to the imperfectness of shape of the shell. If we accept the lower critical load as the working load, then loss of stability of the shell is fully excluded, since the load supported by the shell at post-critical deformation is greater than the lower critical load.

The above-mentioned solution of the working load problem is simple and safe. However, it cannot be accepted because of the very low value of the lower critical load. Let us discuss it, taking as an example a shell having the shape of a shallow spherical segment and subjected to external pressure. For such a shell the value of the upper critical pressure is determined with the help of formula

$$p_e = \frac{2E\delta^2}{\sqrt{3(1-\nu^2)}R^2},$$

where R is the radius of curvature of the shell, and δ is the thickness. The lower critical pressure, p_l, is given by

$$p_l = 3cE\left(\frac{\delta}{R}\right)^2\sqrt{\frac{\delta}{2h}},$$

where h is the height of the segment, and c is a constant $\simeq 0.2$. Hence,

$$\frac{p_l}{p_e} \simeq 0.5\sqrt{\frac{\delta}{2h}}.$$

We note that even at $h=8\delta, p_l/p_e \simeq 0.1$, i.e., the lower critical value equals 0.1 of the upper.

An acceptable solution to the working load problem would be to define such a load, as the load at which loss of stability occurs when initial deflection is taken into account. Presently, we shall attempt to evaluate such a load for the case of a shallow shell of positive Gaussian curvature when subjected to external pressure.

In paragraph 2 we derived the following formula for the value of a load supported by a shell when the bulging rise equals $2h$:

$$p = 3cEH\sqrt{K}\delta^2\sqrt{\frac{\delta}{2h}}.$$

Here, K is the Gaussian and H the mean curvature of the shell. It is natural to assume that if the initial shell deflection is in conformity with the shape of bulging, such a shell will lose its stability when subjected to a pressure given by the above formula. We suggest, therefore, that the critical pressure be evaluated when initial deflection equals $2h$, using the formula

$$p = 3cEH\sqrt{K}\delta^2\sqrt{\frac{\delta}{2h}}, \quad (*)$$

and to consider same as the working pressure.

It should be noted that the derivation of formula (*) was based on the assumption that the parameter $\delta/2h$ is sufficiently small. It follows, therefore, that its application should be limited to such cases when there is a significant initial deflection.

For a spherical shell, formula (*) is reduced to

$$p = 3cE\left(\frac{\delta}{R}\right)^2\sqrt{\frac{\delta}{2h}} = kE\frac{\delta^2}{R^2}.$$

Below, we present a graph (Figure 9), showing the relationship between the nondimensional coefficient k and the initial deflection $2h/\delta$.

This graph shows how doubtful it is to invest in a high degree of shell workmanship, when reliance is placed on the value of the critical pressure which is close to the theoretical.

FIGURE 9.

SUPPLEMENT

LOSS OF STABILITY OF THREE-LAYER SHELLS

The results we have obtained of stability of shells under various modes of loading may be applied to the so-called three-layer shells. A three-layer shell consists of thin exterior layers prepared from materials having high mechanical characteristics, and a comparatively thick layer of the filler made of a weak material. Similarly to the case of ordinary shells, the energy of deformation at bulging of a three-layer shell is concentrated, in the main, at the boundary of the bulging zone and consists of the energy of deformation of the outer layers and the energy of deformation of the filler.

Denoting by u and v the displacements, under deformation, of points of the shell surface in the tangent plane and along the normal respectively, we obtain the following expression for the energy of deformation of the outer layers, per unit length of the boundary of bulging, γ:

$$\bar{U}_e = \frac{\delta E}{1 - \nu^2} \int_{-s^*}^{s^*} \left(\frac{\delta^2 v''^2}{12} + \frac{u^2}{\rho^2} \right) ds,$$

where δ is the thickness of the outer layers; ρ, the radius of curvature of γ; E, the modulus of elasticity; and ν, Poisson's ratio. Integration is performed along the neighborhood of the bulging boundary, γ.

In order to obtain the energy of deformation of the inner layer (the filler), we make the assumption that deformations of the outer layers are identical (Figure 10). As a result of these deformations, the inner layer is subjected to shear deformations, defined by the derivative v', and energy of deformation, per unit volume of the filler, will be

$$\frac{Gv'^2}{2},$$

where G is the shear modulus of the filler. The corresponding energy of deformation of the filler, per unit length of the bulging boundary, will be

$$\bar{U}_i = \frac{Gt}{2} \int_{-s^*}^{s^*} v'^2 \, ds,$$

where t is the thickness of the inner layer.

Consequently, the full energy of deformation, per unit length of a three-layer shell, will be

$$\bar{U}_e + \bar{U}_i = \frac{\delta E}{1 - \nu^2} \int_{-s^*}^{s^*} \left(\frac{\delta^2 v''^2}{12} + \frac{u^2}{\rho_*^2} \right) ds + \frac{Gt}{2} \int_{-s^*}^{s^*} v'^2 \, ds.$$

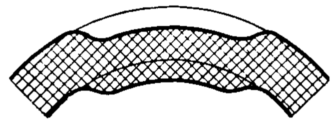

FIGURE 10.

Further, as in the case of ordinary shells considered in section 1, we standardize the variables u, v, s, assuming

$$\bar{u} = \frac{u}{\varepsilon\rho\alpha^2}, \quad \bar{v} = \frac{v'}{\alpha}, \quad \bar{s} = \frac{s}{\rho\varepsilon},$$

where α is the angle between the plane of contact to curve γ and the surface tangent planes, and

$$\varepsilon^4 = \frac{\delta^2}{12\rho^2\alpha^2}.$$

As a result, we obtain the following expression for the energy of deformation:

$$\bar{U}_e + \bar{U}_i = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)} \int\limits_{-\frac{l}{\varepsilon}}^{\frac{l}{\varepsilon}} (\bar{v}'^2 + \bar{u}^2)\, d\bar{s} + \frac{Gt\alpha^2\rho\varepsilon}{2} \int\limits_{-\frac{l}{\varepsilon}}^{\frac{l}{\varepsilon}} \bar{v}^2\, ds.$$

Limiting ourselves to the case of such shells and their deformations for which the parameter $\delta/\rho\alpha$ is small, we change the limits of integration to $\pm\infty$. We then obtain

$$\bar{U}_e + \bar{U}_i = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)} \int\limits_{-\infty}^{\infty} (v'^2 + u^2)\, ds + \frac{Gt\alpha^2\rho\varepsilon}{2} \int\limits_{-\infty}^{\infty} v^2\, ds.$$

For simplification of print, we shall omit the bars above the variables $\bar{u}$, $\bar{v}$, and $\bar{s}$.

We shall determine the shape of the shell at bulging in the zone of strong local bending with the help of the condition of minimum energy under a given general deformation

$$h = -\frac{1}{2\cdot 12^{1/4}}\sqrt{\delta\rho\alpha} \int\limits_{-\infty}^{\infty} v^2\, ds$$

(see section 1). Our problem is therefore reduced to that of minimizing the functional

$$\bar{U}_e + \bar{U}_i$$

when

$$h = \text{const}.$$

We have,

$$\frac{Gt\alpha^2\rho\varepsilon}{2} \int\limits_{-\infty}^{\infty} v^2\, ds = tG\alpha h.$$

It follows that in the case of a three-layer shell, the functional

$$\bar{U}_e + \bar{U}_i$$

differs from the corresponding functional for an ordinary shell by terms which are independent of the variation functions u, v. This means that the functional under consideration has stationary values for the same values of as in the corresponding problem in section 1.

Making use of the result obtained in section 1, we obtain the following expression for energy of deformation of a three-layer shell:

$$\bar{U}_e + \bar{U}_i = \frac{2E\delta^2\alpha^2 h}{\sqrt{12(1-\nu^2)}\,\rho} + tG\alpha h.$$

In order to obtain the full energy of deformation of the shell, we must integrate the above expression along the arc of curve γ, bounding the zone of bulging.

The first term of the expression $\bar{U}_e + \bar{U}_i$ was integrated in section 1. Making use of the results obtained there, we obtain

$$\int_\gamma \bar{U}_e\, ds_\gamma = 2\frac{4\pi E\delta^2}{\sqrt{12(1-\nu^2)}\sqrt{R_1R_2}},$$

where R_1 and R_2 are the main radii of curvature at the center of bulging.

Let us now evaluate

$$\int_\gamma \bar{U}_i\, ds_\gamma.$$

We have (see section 1)

$$\alpha = \rho k_n,$$

where k_n is the normal curvature of shell surface in the γ direction.

$$\frac{1}{\rho} = \frac{\sqrt{R_1R_2}}{\lambda(R_1\sin^2\varphi + R_2\cos^2\varphi)^{3/2}}.$$

$$k_n = \frac{1}{R_1\sin^2\varphi + R_2\cos^2\varphi},$$

$$ds_\gamma = \lambda(R_1\sin^2\varphi + R_2\cos^2\varphi)^{1/2}\,d\varphi,$$

$$\alpha\, ds_\gamma = \frac{\lambda^2(R_1\sin^2\varphi + R_2\cos^2\varphi)\,d\varphi}{\sqrt{R_1R_2}}.$$

It follows that,

$$\int_\gamma \bar{U}_i\, ds_\gamma = Ght\int_\gamma \alpha\, ds_\gamma = \frac{\pi Gt(R_1+R_2)}{\sqrt{R_1R_2}}h\lambda^2.$$

Therefore, the full energy of deformation of the shell is equal to

$$U = \frac{8\pi E\delta^2 h\lambda^2}{\sqrt{12(1-\nu^2)}\sqrt{R_1R_2}} + \frac{\pi Gt(R_1+R_2)}{\sqrt{R_1R_2}}h\lambda^2.$$

The work done by the external pressure, p, equals

$$A = \pi p\sqrt{R_1R_2}h\lambda^2.$$

From the shell equilibrium condition at bulging,

$$d(U-A)=0,$$

where the parameter $h\lambda^2$ is to vary, we obtain the value of the pressure supported by the shell at bulging, i.e., the critical pressure

$$p=\frac{4E\delta^2}{\sqrt{3(1-\nu^2)}\,R_1R_2}+\frac{Gt(R_1+R_2)}{R_1R_2},$$

where R_1 and R_2 are the principal radii of curvature of the shell; δ, the thickness of the outer layers; t, the thickness of the inner layer; E, the modulus of elasticity, and ν, Poisson's ratio of the outer layers, respectively; and G, the shear modulus of the filler.

The formula for p can also be presented in the following form:

$$p=\frac{4E\delta^2K}{\sqrt{3(1-\nu^2)}}+2GtH,$$

where K is the Gaussian, and H the mean curvature of the shell at the center of bulging.

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